Serre Conjecture, Cuntz Algebras and Leavitt Algebras - Preliminary report

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This is a joint work with Roozbeh Hazrat

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- Quillen in USA and, independently, Suslin in Soviet Union (1976): Serre's conjecture holds for any $n \geq 1$.
- Sharma, Ojunguran and Sridharan (1971): Serre's conjecture is false for $K[x_1, \cdots, x_n]$ **if K is a division ring and $n \geq 2$.**

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- As we shall see, this conjecture sits between (i) the algebraic Kirchberg-Phillips problem **[** namely: Two purely infinite simple LPAs of finite graphs $L_K(E) \cong L_K(F) \Leftrightarrow K_0(L_K(E)) \overset{\phi}{\cong} K_0(L_K(F))$ such that $\phi([L_K(E)]) = [L_K(F)]$**]** and

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- We will also interpret the meaning of the Serre conjecture property among graph C*-algebras and show that this property indeed characterizes Cuntz algebras $\mathcal{O}_n$ among graph C*-algebras of finite graphs.

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- A basic reference: **P. Ara, M.A. Moreno and E. Pardo, Non-stable K-theory for graph algebras, Algebra reprsentation Theory, vol. 10 (2007), 157 - 178.**

Ara-Bergman-Moreno-Pardo's Theorem

- Let R be a ring with 1. The set of isomorphism classes of finitely generated projective left R-modules becomes an abelian monoid under the operation $[P] + [Q] = [P \oplus Q]$ and is denoted by $\mathcal{V}(R)$.

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- **Corollary 2**: The V-monoid $\mathcal{V}(L_K(E))$ *is independent of the field K.*

An Easy Criterion

- **Proposition 3:** *Let E be a finite graph and and $L = L_K(E)$. Every finitely generated projective left/right L-module is free if and only if for every $u \in E^0$, there is an integer $k \geq 1$ (depending on u) such that $a_u = k1_E$ in M_E.*

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- Necessity: Let $u \in E^0$. As Lu is free, $[Lu] = [L] + \cdots + [L]$. Then, *in M_E*, Theorem 1 implies $a_u = k1_E$ for some integer k.

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- **Corollary 5**: *If a unital ring* R *is Morita equivalent to* L_n*, then* $R \cong M_d(L_n)$ *for some integer* $d > 0$.
- **Proof**: As f.g. projectives over L_n are free, $R \cong M_d(L_n)$ by Corollary 18.36 in [3], where $d > 0$.

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- **Proof**: Let I be a non-zero order ideal of the monoid M_E. Let $a_u \in I$ for some $u \in E^0$. By Proposition 3, $a_u = k1_E$ for some $k \geq 1$. As I is an order ideal, $k1_E = a_u \in I$ implies that $1_E \in I$. Consequently, $I = M_E$. Since, by Proposition 8, the order ideals of M_E are in bijective correspondence with the graded ideals of $L_K(E)$, we conclude that $L_K(E)$ has no non-zero proper graded two-sided ideals.

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- Recall, a ring R is called **purely infinite simple**, if it is simple and every non-zero one-sided ideal contains an idempotent e such that Re is isomorphic to a proper direct summand of itself.
- Leavitt proved that the Leavitt ring $L(1, n)$ is purely infinite simple.

Theorem 11: *Let E be a finite graph. Let $L := L_K(E)$. If every finitely generated projective left/right L-module is free, then L is one of the following:*
1. $L \cong K$;
2. $L \cong K[x, x^{-1}]$;
3. L is **purely infinite simple** *with $L = < c^0 >$ where c is an extreme cycle (and thus E* contains cycles with exits, *E^0 is downward directed, contains no non-empty proper hereditary saturated subsets and every $u \in E^0$ connects to a $w \in c^0$). Further, for some positive integer n, $K_0(L) \cong K_0(L(1, n+1))$.* Moreover,

$$(K_0(L), [L]) \cong (\mathbb{Z}/n\mathbb{Z}, 1) \cong (K_0(L(1, n+1)), [L(1, n+1)])$$

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- So assume that the graph E is neither a single vertex nor a single loop. This means that either E has at least two distinct vertices or E has a single vertex u at which two or more loops are based. In the latter case we are done since E will be the Rose graph R_m and $L = L_K(R_m)$ will be purely infinite simple. So assume that $|E^0| \geq 2$.

- Now, by Proposition 3, to each vertex v in E there is an integer $k > 0$ such that $a_v = k1_E$ in $M_{E.}$. Suppose w is a sink in E. In M_E, $a_w \neq 1_E$ and, as w does not emit any edges, $a_w \neq na_w$ for every positive integer $n > 1$. This is a contradiction, since, by supposition, there is a positive integer k such that $k1_E = a_w$ which would imply that $ka_w = a_w$. Likewise, suppose, there is a cycle $c = e_1 \cdots e_n$ without exits, where $n > 1$ and $s(e_i) = v_i$. Since each v_i emits exactly one edge, we have in M_E, the relation $a_{v_1} = a_{v_2} = \cdots = a_{v_n}$. Since this is the only relation involving a_{v_1}, $a_{v_1} \neq kv_1$ for any integer $k > 1$. Again this contradicts our supposition that $a_{v_1} = k1_E$ (where $k > 1$, as $a_{v_1} \neq 1_E$) which implies that $a_{v_1} = ka_{v_1}$ for some $k > 1$.

- **Step 3**: Thus the finite graph E contains no sinks, and no cycles without exits. By Lemma 9, E has no non-empty proper hereditary saturated subsets of vertices. Further, by Lemma 10, every vertex in E will connect to some extreme cycle in E. Thus cycles in E have exits and we conclude that L is a simple ring. Let c be an extreme cycle in E. As $L = L_K(E)$ is graded-simple (Lemma 9), $L =< c^0 >$. By Theorem 3.7.6 in **[AAS]**, L is purely infinite simple .

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- **Step 4**: Consider the additive map $\phi : \mathbb{N} \longrightarrow M_E$ such that

  $1 \longmapsto 1_E$. Since, for every $v \in E^0$, $a_v = k1_E$, ϕ is an epimorphism. Also observe that $1_E = n1_E$ for some integer $n > 1$. To see this, let $E^0 = \{v_1, \cdots, v_m\}$ (where $m > 1$) so that $1_E = a_{v_1} + \cdots + a_{v_m}$. Since for i, $a_{v_i} = k_i 1_E$, substituting for the a_{v_i} , we get $1_E = n1_E$ for some integer $n > 1$. Since $M_E \cong \mathcal{V}(L(E))$, ϕ gives rise to an epimorphism $\bar{\phi} : \mathbb{Z} \longrightarrow K_0(L(E))$ under which $1 \longmapsto [L_K(E)]$. Consequently, $K_0(L(E)) = \bar{\phi}(\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ for some positive integer n. It is known [1] that there is an isomorphism $K_0(L(1, n+1)) \longrightarrow \mathbb{Z}/n\mathbb{Z}$ mapping $[L(1, n+1)] \longmapsto 1$. Thus $(K_0(L), [L]) \cong (\mathbb{Z}/n\mathbb{Z}, 1) \cong (K_0(L(1, n+1)), [L(1, n+1)])$.

- **Statement-1: The algebraic Kirchberg-Phillips problem:** Let E, F be finite graphs such that $L_K(E), L_K(F)$ are purely infinite simple. Then $L_K(E) \cong L_K(F)$ if and only if there is an isomorphism $\phi : K_0(L_K(E)) \longrightarrow K_0(L_K(F))$ such that $\phi([L_K(E)]) = [L_K(F)]$.

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- **Statement-2:** If E is a finite graph, then finitely generated projective modules over $L_K(E)$ are free $<\mathbf{=}>$ $L_K(E) \cong L_n := L(1, n)$.
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- That Statement-1 $=>$ Statement-2 is immediate from the last part of Theorem 11 (3).
- To prove Statement-2 $=>$ Statement-3: Now $L_{2-} \cong L_K(F)$ where F is the Cuntz splice graph $u \cdot \overset{\circlearrowleft}{\underset{\circlearrowleft}{}} \rightleftarrows v \cdot \circlearrowleft \rightleftarrows w \cdot \circlearrowleft$. By a direct computation, one can show that the monoid M_F consists of exactly two elements, which then necessarily must be $\{0\}$ and $[1_{L_K(F)}]$. So then clearly finitely generated projective modules over L_{2-} are free. Then, by Statement-2, $L_{2-} \cong L_m$ for some m. We claim that $m = 2$. Because, for $m > 2$, $K_0(L_m) \cong \mathbb{Z}/(m-1)\mathbb{Z}$ while $K_0(L_{2-}) = 0 = K_0(L_2)$. Thus $L_{2-} \cong L_2$.

Remark: The graded version of the Serre's conjecture property has a negative answer: If E is a finite graph and if finitely generated graded projectives over $L_K(E)$ are graded free, then $L_K(E)$ need not be isomorphic to L_n. Take E to be the graph $\cdot \circlearrowleft \quad \rightleftarrows \quad \cdot$. A talented monoid argument (by Roozbeh) shows that $L_K(E) \ncong L_n$.
A natural question is: Which Leavitt path algebras $L_K(E)$ of a finite graph E have the graded Serre conjecture property?

Preliminaries of C*-Algebras

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- An important result is the following.
- **Theorem 12**: ([2], Theorem 7.1) Let E be a finite graph. *The natural inclusion $L_{\mathbb{C}}(E) \longrightarrow C^*(E)$ induces an isomorphism $\mathcal{V}(L_{\mathbb{C}}(E)) \longrightarrow \mathcal{V}(C^*(E))$.*

- How to interpret the Serre's conjecture for graph C*-algebras where the concepts of projective and free modules are not formally considered in the definition of V-monoid? Motivated by Proposition 3, we take the monoidal approach to address this question.

Serre's Conjecture Property for C*-Algebras

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- **Definition**: Let E be a finite graph. We say that the **Serre's conjecture property holds in** $C^*(E)$ if for each $v \in E^0$, there is a positive integer k such that $a_v = k1_E$ in M_E.

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- **Theorem 13: Let** E be any finite graph which is not just a single vertex or just a single loop. Then the C*-algebra $C^*(E)$ has the Serre's conjecture property if and only if, *for some* $n > 0$, $C^*(E)$ is isomorphic to t*he Cuntz algebra* $\mathcal{O}_{n+1}$.

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- Follow the proof of the sufficiency part of Proposition 3 to conclude that the Serre's conjecture property holds in $C^*(R_n) = \mathcal{O}_n$.

- Necessity: Suppose the Serre's conjecture property holds in $C^*(E)$. Since for each $v \in E^0$, there is a positive integer k such that $a_v = k1_E$ in M_E, repeating the proof of Lemma 9, we conclude that that $C^*(E)$ is *"gauge-invariant simple"*, that is, it has no non-zero proper gauge-invariant ideals. Then, repeating the proof of Theorem 11, we conclude that $C^*(E)$ is purely infinite simple and that $K_0(C^*(E)) \cong \mathbb{Z}/n\mathbb{Z}$ for some positive integer n. As $K_0(\mathcal{O}_{n+1}) \cong \mathbb{Z}/n\mathbb{Z}$, we conclude that $(K_0(C^*(E)), [C^*(E)]_0) \cong (\mathbb{Z}/n\mathbb{Z}, 1) \cong (K_0(\mathcal{O}_{n+1}), [\mathcal{O}_{n+1}]_0)$. Now E is a finite graph without sinks and $K_0(C^*(E)) \cong K_0(\mathcal{O}_{n+1})$. Then, by Tomforde ([5]), $K_1(C^*(E)) \cong K_1(\mathcal{O}_{n+1})$. We then apply the Kirchberg-Phillips theorem for graph C*-algebras (see Theorem 2.3.28, [5]) to conclude that $C^*(E) \cong \mathcal{O}_{n+1}$.

- G. Abrams, P. Ara and M. Siles Molina, Leavitt Path Algebras (book)
- P. Ara, M.A. Moreno and E. Pardo, Non-stable K-theory for graph algebras, Algebra representation Theory, vol. 10 (2007), 157 - 178.
- T.Y. Lam, Lectures on modules and rings, GTM vol. 189, Springer (1999)
- R. Hazrat and K.M. Rangaswamy, On Graded Irreducible Representations of Leavitt Path Algebras, J. Algebra vol. 450 (2016), 458 - 486.
- M. Tomforde, Structure of graph C*-algebras and generalizations, Graph Algebras - Bridging the gap between analysis and algebra, Proceeding of the workshop in graph algebras, University de Malaga (2007).