# An Abstract Characterization for Projections in Operator Systems

Rings and Wings: ARCS Center

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# Some Preliminaries

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The examples to keep in mind will be $B(\ell_2^n)$ (i.e. $M_n$), $B(\ell_2)$, and $B(H)$ for an abitrary Hilbert space $H$.
Every $C^*$-algebra "is" a closed subalgebra of $B(H)$ for some Hilbert space $H$. (Gelfand-Naimark)

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### Definition

A *concrete operator system* is a self-adjoint unital subspace $\mathcal{V} \subset B(H)$. Abstractly, an operator system is a triple $(\mathcal{V}, \{C_n\}_{n\in\mathbb{N}}, e)$ where $\mathcal{V}$ is a $*$-vector space, $\{C_n\}_{n\in\mathbb{N}}$ is a proper matrix ordering on $\mathcal{V}$ and $e$ is an Archimedean matrix order unit.

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If for each $n \in \mathbb{N}$ it follows $C_n \cap -C_n = \{0\}$ then we say the collection $\{C_n\}_{n\in\mathbb{N}}$ is a *proper matrix ordering*. Our use of the term differs just a bit from convention.

An element $e \in \mathcal{V}$ is called a *matrix order unit* if for any $v \in M_n(\mathcal{V})_h$ there exists an $r > 0$ such that $re_n - v \in M_n(\mathcal{V})^+$. (Here we have let $e_n := I_n \otimes e$). $e$ is called an *Archimedean matrix order unit* if we have the additional property that if $v \in M_n(\mathcal{V})$ and for all $\epsilon > 0$ it follows $\epsilon e_n + v \in M_n(\mathcal{V})^+$ then $v \in M_n(\mathcal{V})^+$.

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## Theorem ([CE77])

*Given any abstract operator system $\mathcal{V}$ there exists a Hilbert space and a concrete operator system $\mathcal{W} \subset B(H)$ such that $\mathcal{V} \simeq \mathcal{W}$.*

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Here (and through the rest of the talk), we use $\simeq$ to denote a complete order isomorphism of operator systems, i.e., there exists a bijection $\varphi : \mathcal{V} \to \mathcal{W}$ such that $\varphi$ and $\varphi^{-1}$ are both completely positive.
*Completely positive* means that given a linear map $\varphi : \mathcal{V} \to \mathcal{W}$ then for all $n \in \mathbb{N}$ the induced *nth amplification* $\varphi_n : M_n(\mathcal{V}) \to M_n(\mathcal{W})$ defined by

$$\varphi_n(v) := \sum_{ij} e_i e_j^* \otimes \varphi(v_{ij}), v \in M_n(\mathcal{V}) \tag{1}$$

is completely positive.

# Why would we care about operator systems (or operator spaces for that matter)

For me the motivation comes from tensor products. Given two $C^*$-algebras $\mathcal{A}_i, i = 1, 2$ one may from the algebraic tensor product $\mathcal{A}_1 \otimes \mathcal{A}_2$ and consider various $C^*$-algebra structures on this tensor product.

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$$\mathcal{A}_1 \otimes_{\mathsf{min}} \mathcal{A}_2,$$
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# The Concrete Motivation

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Consider a concrete operator system $\mathcal{V} \subset B(H)$ and an element $p \in \mathcal{V}^+$ which is a projection when viewed as an operator in $B(H)$. Letting $p_n := I_n \otimes p$, consider the following collection of sets $\{C(p_n)\}_{n\in\mathbb{N}}$ where

$$C(p_n) := \{x \in M_n(\mathcal{V}) : x = x^*, p_n x p_n \in B(H^n)^+\}. \qquad (2)$$

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## Proposition (AR)

*Let $\mathcal{V} \subset B(H)$ be an operator system and suppose that $p \in \mathcal{V}$ where $p$ is a projection in $B(H)$. The sequence of sets $\{C(p_n)\}_n$ is a matrix ordering on $\mathcal{V}$. Furthermore If $p \leq q \leq I$ then $q$ is an Archimedean matrix order unit for $(\mathcal{V}, \{C(p_n)\}_n)$.*

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- Let $\alpha_o : \mathcal{V} \to [0,\infty)$ denote the order semi-norm induced by the projection $p$. Then the cones $C(p_n)$ are $\alpha$-closed.
- Fix $n \in \mathbb{N}$ and let $J_{p_n} := \operatorname{span} C(p_n) \cap -C(p_n)$. Then $M_n(J_p) = J_{p_n}$.

We now wish to consider the quotient $*$-vector space $\mathcal{V}/J_p$ where $J_p := \operatorname{span} C(p) \cap -C(p)$. For each $n \in \mathbb{N}$ let

$$\widetilde{C}(p_n) := \{(x_{ij} + J_p)_{ij} \in M_n(\mathcal{V}/J_p) : x = (x_{ij})_{ij} \in C(p_n)\} \quad (3)$$

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## Definition

Given an operator system $\mathcal{V} \subset B(H)$ with $p \in B(H)$ a projection, we call the set $p\mathcal{V}p$, regarded as linear operators on the Hilbert space $pH$, the *concrete compression operator system*.

Keep the following compression operator system in mind: with $\mathcal{V} \subset B(H)$ and $p \in \mathcal{V}^+$ as before, let $q = I - p$. Of particular interest to us is the abstract analogue of the compression operator system

$$(p \oplus q)M_2(\mathcal{V})(p \oplus q).$$

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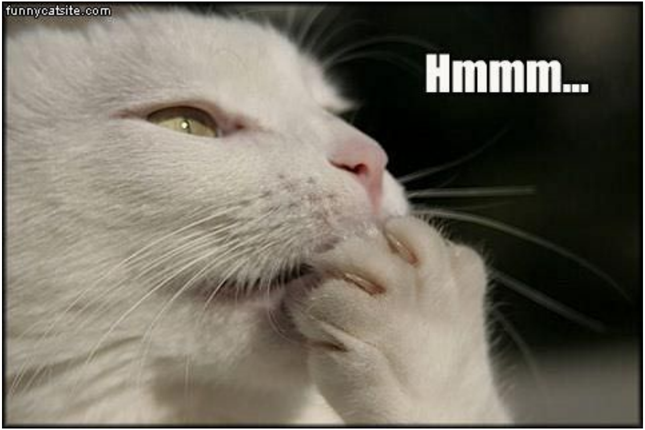

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3. Consider the collection $\{\widetilde{C}_n\}_n$ such that for each $n \in \mathbb{N}$,

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In particular, we have the following:

## Proposition (AR)

*Suppose that $\mathcal{V}$ is a $*$-vector space with matrix ordering $\{C_n\}_n$ and an Archimedean matrix order unit $e$. Then $(\mathcal{V}/J, \{\widetilde{C}_n\}_n, e + J)$ is an operator system.*

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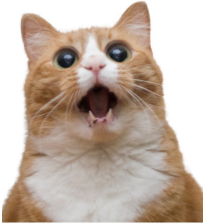

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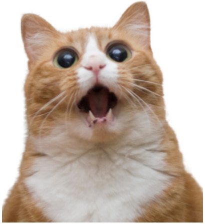

## Lemma (AR)

*Let $\mathcal{V} \subset B(H)$ be an operator system and suppose that $p \in \mathcal{V}$ is a projection. Then for any $x \in \mathcal{V}$ with $x = x^*$, we have that $pxp \geq 0$ in $B(H)$ if and only if for every $\epsilon > 0$ there exists a $t > 0$ such that*

$$x + \epsilon p + t(I - p) \geq 0.$$

## Definition

Let $(\mathcal{V}, \{C_n\}_n, e)$ be an operator system, and suppose that $p \in \mathcal{V}$ with $0 \leq p \leq e$, i.e., let $p \in \mathcal{V}$ be a positive contraction of $\mathcal{V}$. For each $n \in \mathbb{N}$ and let $p_n = I_n \otimes p$. We define the *positive cone relative to* $p_n$, denoted $C(p_n)$, to be

$$C(p_n) := \{x \in M_n(\mathcal{V}) : x = x^*, \text{ for all } \epsilon > 0 \text{ there exists } t > 0 \tag{5}$$

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$$\text{such that } x + \epsilon p_n + t(e_n - p_n) \in C_n\}. \tag{6}$$

An immediate consequence of the previous lemma is that if a positive contraction $p \in \mathcal{V}$ is a projection, then for each $n \in \mathbb{N}$ the positive cone relative to $p_n$ becomes

$$C(p_n) = \{x \in M_n(\mathcal{V}) : x = x^*, p_n x p_n \in B(H^n)^+\}, \tag{7}$$

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$$\alpha_m(x) = \sup\{|\varphi(x)| : \varphi \in \mathcal{S}(\mathcal{V})\} \tag{8}$$

where $\mathcal{S}(\mathcal{V})$ denotes the set of states on $\mathcal{V}$. It is not difficult to show that if $\alpha_o : \mathcal{V}_h \to [0, \infty)$ denotes the order norm induced by $e$ given by

$$\alpha_o(x) = \inf\{t > 0 : te \pm x \in \mathcal{V}^+\}, \tag{9}$$

then $\alpha_o = \alpha_m$ when restricted to $\mathcal{V}_h$. (Paulsen and Tomforde [PT09])

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## Proposition (AR)

*Let $\mathcal{V}$ an operator system and let $p \in \mathcal{V}$ be a nonzero positive contraction. Let $\alpha_m : \mathcal{V} \to [0, \infty)$ denote the minimal order norm induced by $e$. Then $\alpha_m(p) = 1$ if and only if $p \notin J_p$.*

As in our previous proposition we will define the family of sets $\{\widetilde{C}(p_n)\}_n$ where for each $n \in \mathbb{N}$ we have

$$\widetilde{C}(p_n) = \{(x_{ij} + J_p) \in M_n(\mathcal{V}/J_p) : x = (x_{ij}) \in C(p_n)\}. \tag{10}$$

We now have the abstract analogue to theorem on concrete compressions of operator systems:

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## Theorem (AR)

*Given an operator system $\mathcal{V}$ and positive contraction $p \in \mathcal{V}$ such that $\alpha_m(p) = 1$, the triple*

$$(\mathcal{V}/J_p, \{\widetilde{C}(p_n)\}_{n\in\mathbb{N}}, p + J_p)$$

*is a non-trivial operator system.*

As in our previous proposition we will define the family of sets $\{\widetilde{C}(p_n)\}_n$ where for each $n \in \mathbb{N}$ we have

$$\widetilde{C}(p_n) = \{(x_{ij} + J_p) \in M_n(\mathcal{V}/J_p) : x = (x_{ij}) \in C(p_n)\}. \tag{10}$$

We now have the abstract analogue to theorem on concrete compressions of operator systems:

## Theorem (AR)

*Given an operator system $\mathcal{V}$ and positive contraction $p \in \mathcal{V}$ such that $\alpha_m(p) = 1$, the triple*

$$(\mathcal{V}/J_p, \{\widetilde{C}(p_n)\}_{n\in\mathbb{N}}, p + J_p)$$

*is a non-trivial operator system.*

## Corollary

*Suppose that $\mathcal{V} \subset B(H)$ is an operator system and that $p \in \mathcal{V}$ is a projection in $B(H)$. Then the abstract compression $(\mathcal{V}/J_p, \{\tilde{C}(p_n)\}_{n\in\mathbb{N}}, p + J_p)$ is completely order isomorphic to the concrete compression $p\mathcal{V}p$.*

### Definition

Given an operator system $\mathcal{V}$ and a positive contraction $p \in \mathcal{V}$ such that $\alpha_m(p) = 1$ then we call the operator system $(\mathcal{V}/J_p, \{\widetilde{C}(p_n)\}_{n\in\mathbb{N}}, p + J_p)$ the *abstract compression operator system* and denote it by $\mathcal{V}/J_p$.

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Though we do not use it here a nice corollary to the previous theorem is that we may consider compressions of operator systems by finite families of positive contractions where at least one has minimal norm 1.

Where are those abstract projections you talked about?????

# Abstract Projections

# Abstract Projections

## Definition

Let $(\mathcal{V}, \{C_n\}_n, e)$ be an abstract operator system and suppose that $0 \leq p \leq e$ for some $p \in \mathcal{V}^+$ and $\alpha_m(p) = 1$. Set $q = e - p$. We call $p$ an *abstract projection* if the map $\pi_p : \mathcal{V} \to M_2(\mathcal{V})/J_{p \oplus q}$ defined by

$$\pi_p : x \mapsto \begin{pmatrix} x & x \\ x & x \end{pmatrix} + J_{p \oplus q}$$

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This leads to a representation theorem.

## Proposition (AR)

*Suppose that $\mathcal{V}$ is an operator system and that $p$ is an abstract projection in $\mathcal{V}$. Let $q = e - p$. Then for every ucp map $\phi : M_2(\mathcal{V})/J_{p\oplus q} \to M_n$ there exists a $k \in \mathbb{N}$ and a ucp map $\psi : M_2(\mathcal{V})/J_{p\oplus q} \to M_k$ such that $\psi(p \oplus 0 + J_{p\oplus q})$ and $\psi(0 \oplus q + J_{p\oplus q})$ are projections and satisfying the property that*

$$\phi_{2n}\left(\begin{pmatrix} a & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b^* & 0 & 0 & c \end{pmatrix} + M_{2n}(J_{p\oplus q})\right)$$

$$= \begin{pmatrix} \phi_n\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + M_n(J_{p\oplus q})\right) & \phi_n\left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} + M_n(J_{p\oplus q})\right) \\ \phi_n\left(\begin{pmatrix} 0 & 0 \\ b^* & 0 \end{pmatrix} + M_n(J_{p\oplus q})\right) & \phi_n\left(\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} + M_n(J_{p\oplus q})\right) \end{pmatrix}$$

$$\geq 0$$

*if and only if* $\psi_n\left(\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} + M_n(J_{p\oplus q})\right) \geq 0$ *for all* $a, b, c \in M_n(\mathcal{V})$.

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- The map $\psi : M_2(\mathcal{V})/J_{p\oplus q} \to M_k$ is defined by

$$\begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix} \begin{pmatrix} \phi\widehat{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}} & \phi\widehat{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}} \\ \phi\widehat{\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}} & \phi\widehat{\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}} \end{pmatrix} \begin{pmatrix} V^* & 0 \\ 0 & W^* \end{pmatrix}.$$

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- $\psi$ is necessarily unital by construction. To show complete positivity show that $\phi$ is supported in the proper corners.

### Theorem (AR)

*Suppose that $\mathcal{V}$ is an operator system and that $p \in \mathcal{V}$ is an abstract projection. Then there exists a unital complete order embedding $\pi : \mathcal{V} \to B(H)$ such that $\pi(p)$ is a projection in $B(H)$.*

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- Show that this new direct sum is a complete order embedding. (Unitality comes from construction).

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One may compare this theorem with a result of Blecher and Neal [BN11]. They proved that given a unital operator space $(\mathcal{E}, u)$ then $u$ is necessarily a unitary in the ternary envelope $T(\mathcal{E})$.

# Some Remarks on Correlation Sets

Let $n, k \in \mathbb{N}$. The tuple $\{p(a, b|x, y) : x, y \in [n], a, b \in [k]\}$ is called a *correlation* if

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Here we refer to the number $n$ as the number of experiments and $k$ as the number of outcomes.

Suppose for each $x \in [n]$ and $a \in [k]$ the quantity

$$p_A(a|x) := \sum_b p(a, b|x, y) \tag{11}$$

is well-defined. Similarly assume for each $y \in [n]$ and $b \in [k]$ the quantity

$$p_B(b|y) := \sum_a p(a, b|x, y) \tag{12}$$

is well-defined.

If the correlation $p(a, b|x, y)$ satisfies these properties then we call it *non-signalling* and denote the set of all such correlations as $C_{ns}(n, k)$. The non-signalling conditions model that Alice and Bob perform their experiments independently without talking to one-another.

# PVMS

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$$C_{loc}(n,k) \subset C_q(n,k) \subset C_{qs}(n,k) \subset C_{qa}(n,k) \subset C_{qc}(n,k) \subset C_{ns}(n,k)$$

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Operator system techiniques were used in studying correlation sets. (see e.g. [Pau+16; Lup+20])
Tsirelson's conjecture in fact does not hold as recently shown in [Ji+20].
Though sharp bounds on $(n,k)$ are not known.

Let us take a look at the following:

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## Proposition

*Let $n$ and $k$ be positive integers. Then the following statements are equivalent.*

1. $\{p(a,b|x,y)\} \in C_{qc}(n,k)$ *(resp. $\{p(a,b|x,y)\} \in C_q(n,k)$).*
2. *There exists a (resp. finite dimensional) C*-algebra $\mathcal{A}$, projection valued measures $\{E_{x,a}\}_{a=1}^k, \{F_{y,b}\}_{b=1}^k \subset \mathcal{A}$ for each $x, y \leq n$ satisfying $E_{x,a}F_{y,b} = F_{y,b}E_{x,a}$ for all $x, y \leq n$ and $a, b \leq k$, and a state $\phi : \mathcal{A} \to \mathbb{C}$ such that $p(a,b|x,y) = \phi(E_{x,a}F_{y,b})$.*
3. *There exists an operator system $\mathcal{V} \subset B(H)$ (resp. for a finite dimensional Hilbert space $H$), projection valued measures $\{E_{x,a}\}_{a=1}^k, \{F_{y,b}\}_{b=1}^k$ for each $x, y \leq n$ satisfying $E_{x,a}F_{y,b} \in \mathcal{V}$ and $E_{x,a}F_{y,b} = F_{y,b}E_{x,a}$ for all $x, y \leq n$ and $a, b \leq k$, and a state $\phi : \mathcal{V} \to \mathbb{C}$ such that $p(a,b|x,y) = \phi(E_{x,a}F_{y,b})$.*

## Definition

Let $n, k \in \mathbb{N}$. We call an operator system $\mathcal{V}$ a *non-signalling* operator system if it is the linear span of positive operators $\{Q(a, b|x, y) : a, b \leq k, x, y \leq n\} \subset \mathcal{V}$, called the *generators* of $\mathcal{V}$, with the properties that $\sum_{a,b} Q(a, b|x, y) = e$ for each choice of $x, y \leq n$ and that the operators

$$E(a|x) := \sum_{b} Q(a, b|x, y)$$

and

$$F(b|y) := \sum_{a} Q(a, b|x, y)$$

are well-defined (i.e. $E(a|x)$ is independent of the choice of $y$ and $F(b|y)$ is independent to the choice of $x$).

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We call an operator system $\mathcal{V}$ a *quantum commuting* operator system if it is a non-signalling operator system with the property that each generator $Q(a, b|x, y)$ is an abstract projection in $\mathcal{V}$.

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## Theorem (AR)

*A correlation* $\{p(a,b|x,y)\}$ *is non-signalling (resp. quantum commuting) if and only if there exists a non-signalling (resp. quantum commuting) operator system* $\mathcal{V}$ *with generators* $\{Q(a,b|x,y)\}$ *and a state* $\phi : \mathcal{V} \to \mathbb{C}$ *such that* $p(a,b|x,y) = \phi(Q(a,b|x,y))$ *for each* $a,b,x,y$*.*


I DON'T ALWAYS
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BUT WHEN I DO, YOU'RE
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