The Graded Classification Conjecture for graph algebras

context, some progress, current status

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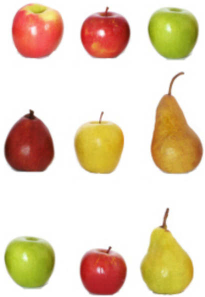


classification

Everybody has their favorite conjecture

The one we shall talk about today...

The Graded Classification Conjecture

Conceived by Roozbeh.

Contemplated by others...

Grothendieck group of a ring

A ring R

Start with the fin. gen. projectives.

Look at the monoid $\mathcal{V}(R)$ of their iso classes with

$$[P]+[Q]=[P\oplus Q].$$

Its K_0-group

Force the cancellativity,

complete to a group. Get the Grothendieck group

$$K_0(R).$$

A general question...

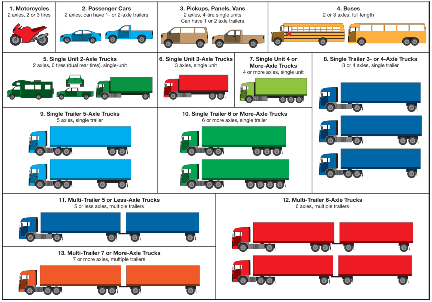
How well $K_0(R)$ reflects the properties of R?

$R \cong S$ as rings iff $K_0(R) \cong K_0(S)$ as (pointed) groups?

$\Rightarrow$ always holds.

$\Leftarrow$ "rarely" holds.

E.g. R, S are matricial algebras over a field.



Classification

Let us focus on LPAs

For LPAs, $\Leftarrow$ <u>really</u> rarely holds.

$E = \bullet$ 

$F = \bullet$ 

$L_K(E) \ncong L_K(F)$ but $K_0(L_K(E)) = K_0(L_K(F)) = \mathbb{Z}$

Also, for $E = \bullet$ 

$L_K(E) \ncong 0$ but $K_0(L_K(E)) = K_0(0) = 0.$

How to compute K_0 of a LPA?

Starting from a (row-finite) graph E, define a monoid M_E, called the **graph monoid**, generated by the elements $[v]$ (think the iso class of $L_K(E)v$) where v is a vertex, subject to the relation

$$[v] = \sum_{e \in \boldsymbol{s}^{-1}(v)} [\boldsymbol{r}(e)]$$

if v is regular.

Why? Because left multiplication by e is an iso of fin. gen. proj. $\boldsymbol{r}(e)L_K(E) = e^*eL_K(E)$ and $ee^*L_K(E)$. So, if v is regular, then

$$[v] = \left[\sum_{e \in \boldsymbol{s}^{-1}(v)} ee^*\right] = \sum_{e \in \boldsymbol{s}^{-1}(v)} [e^*e] = \sum_{e \in \boldsymbol{s}^{-1}(v)} [\boldsymbol{r}(e)].$$

M_E and G_E

Then form the Grothendieck group G_E of M_E and we have that

$$\boxed{\begin{array}{c} M_E \cong \mathcal{V}(L_K(E)) \\ G_E \cong K_0(L_K(E)) \end{array}}$$

For example, if $E = \bullet^v$, then

$M_E = \langle v | v = v + v \rangle$. Its Grothendieck group is trivial since $v = v + v \Rightarrow 0 = v$.

Enter the grading to the rescue!

Leavitt path algebra is also **graded**.

If Γ is a group, a ring R is Γ-**graded** if

$$R = \bigoplus_{\gamma \in \Gamma} R_\gamma \quad \text{such that} \quad R_\gamma R_\delta \subseteq R_{\gamma\delta}.$$

In the world of graded rings...

... "**element**" is replaced by "**homogeneous element**" ($x \in R_\gamma$ for some γ) and "**module**" by "**graded module**".

ring

graded ring

Many rings are **naturally graded**: group rings, LPAs ...

For a LPA, $\Gamma = \mathbb{Z}$ and $L_K(E)_n = \text{span}\,\{pq^* \mid |p| - |q| = n\}$.

Shifts, graded free modules, graded matrix rings

A module M is **graded** if $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ such that $R_\gamma M_\delta \subseteq M_{\gamma\delta}$.

Every graded module M can be **shifted** by δ as follows.

$$M(\delta) = \bigoplus_{\gamma \in \Gamma} M_{\gamma\delta} \quad \text{so that} \quad M(\delta)_\gamma = M_{\gamma\delta}.$$

A finitely generated **graded free** R-module is of the form

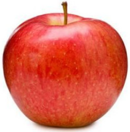
$$R(\gamma_1) \oplus \ldots \oplus R(\gamma_n).$$

$\mathbb{M}_n(R)(\gamma_1, \ldots, \gamma_n)$ is $\mathbb{M}_n(R)$ with a Γ-grading so that

$$\mathbb{M}_n(R)(\gamma_1, \ldots, \gamma_n) \quad \cong_{\mathrm{gr}} \quad \mathrm{End}_R \left(\bigoplus_{i=1}^n R(\gamma_i^{-1}) \right)$$

Finitely generated graded free modules

If $\Gamma =$ trivial, and K is a field, there is **just one one-dimensional free module**: K.



If $\Gamma = \mathbb{Z}$, for example, and R is Γ-graded there can be

many one-dimensional graded free modules:

$\ldots R(-3),\ R(-2),\ R(-1),\ R(0),\ R(1),\ R(2),\ R(3), \ldots$

Three examples

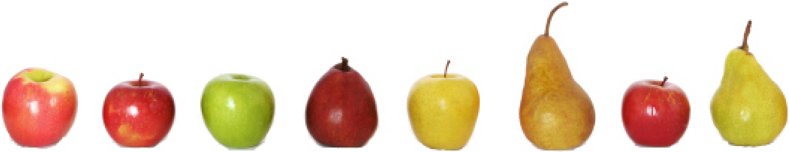
Let $\Gamma = \langle x \rangle \cong \mathbb{Z}$ and $R = K[x, x^{-1}] = K[\mathbb{Z}]$.

1. Let us grade R **trivially**, i.e.

$$\boxed{\begin{array}{ll} K[x,x^{-1}]_0 = K[x,x^{-1}] & \\ K[x,x^{-1}]_n = 0 & n \neq 0 \end{array}}$$

then

$$R(m) \not\cong_{\mathrm{gr}} R(n)$$

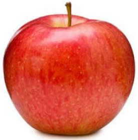


$\dots, R(-3), R(-2), R(-1), R(0), R(1), R(2), R(3), R(4), \dots$

The second two examples

2. Same Γ, same R. Let us grade R by

$$\boxed{K[x,x^{-1}]_n = K\{x^n\}}$$ then



$$R(m) \cong_{\mathrm{gr}} R(n)$$

3. $E =$
$$\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$
$F =$
$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

$L_K(E) \cong L_K(F) \cong \mathbb{M}_3(K)$ as algebras. However,

$$L_K(E) \cong_{\mathrm{gr}} \mathbb{M}_3(K)(0,1,2) \not\cong_{\mathrm{gr}} L_K(F) \cong_{\mathrm{gr}} \mathbb{M}_3(K)(0,1,1)$$

as graded algebras.

"Graded" version of the K_0-group

If R is Γ-graded, replace **"projective"** by **"graded projective"** and repeat the construction for $\mathcal{V}(R)$, get $\mathcal{V}^\Gamma(R)$ with the Γ-action induced by the shifts.

$$\gamma[P] = [P(\gamma)].$$

Then get the **Grothendieck Γ-group** $K_0^\Gamma(R)$. Roozbeh calls it the graded Grothendieck group and uses $K_0^{\mathrm{gr}}(R)$.

K_0^Γ classifies better

K_0^Γ of a Γ-graded ring is a $\mathbb{Z}[\Gamma]$-module. Because of this additional structure,

K_0^Γ is **more sensitive** than K_0.

So, it classifies better.

For LPAs, $\Gamma = \langle x \rangle \cong \mathbb{Z}$, $\mathbb{Z}[\Gamma]$ is $\mathbb{Z}[x, x^{-1}]$ ($\mathbb{Z}[\mathbb{Z}]$)

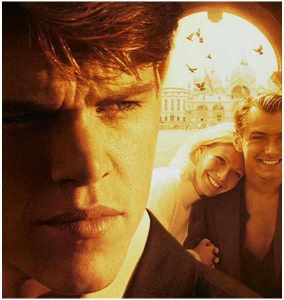
$K_0^\Gamma(L_K(E))$ is a $\mathbb{Z}[x, x^{-1}]$-module.

Graph-only approach

Recall that M_E is defined using a graph E only and G_E is its Grothendieck completion. We want the Γ-versions.

For a group Γ and a graph E, one wants a monoid M_E^Γ,

the graph Γ-monoid.



Roozbeh and Huanhuan (Li) call it **the talented monoid.**

The Grothendieck group of M_E^Γ is the **the graph Γ-group** G_E^Γ.

$$M_E^\Gamma \cong \mathcal{V}^\Gamma(L_K(E))$$
$$G_E^\Gamma \cong K_0^\Gamma(L_K(E))$$

Computing M_E^Γ and G_E^Γ

Let us concentrate on $\Gamma = \langle x \rangle$.

M_E^Γ has the same generators $[v]$ as M_E but the defining relation is modified by adding just one x in the formula from before. It becomes

$$\boxed{[v] = \sum_{e \in \boldsymbol{s}^{-1}(v)} x[\boldsymbol{r}(e)]}$$

for all regular vertices v.

Why x?
Because $x = x^1$ is the length of the path e from v to $\boldsymbol{r}(e)$.

Two examples

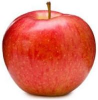
$E = \bullet$ 

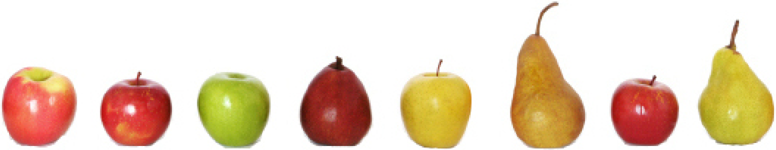
$F = \bullet$ 

$$M_E^\Gamma = \mathbb{Z}^+[x, x^{-1}]$$
$$G_E^\Gamma = \mathbb{Z}[x, x^{-1}]$$

$M_F^\Gamma = \mathbb{Z}^+$ with trivial Γ-action.

$G_F^\Gamma = \mathbb{Z}$ with trivial Γ-action.

Just like in the examples with  and



One more example

Let us compare M_E and M_E^Γ

for the rose $\quad E = \bullet^v$

- $M_E = \langle v \mid v = v + v \rangle$ so $G_E = 0$.
- $M_E^\Gamma = \langle v \mid v = xv + xv \rangle$ and G_E^Γ is isomorphic to $\mathbb{Z}[\frac{1}{2}]$ if we identify v with 1 and the action of x by multiplication by $\frac{1}{2}$.

In general, M_E^Γ is **cancellative**.

All seems so good...

... that Roozbeh formed the following question (circa 2011):

Is for any two graphs E and F,

$$L_K(E) \cong_{\mathrm{gr}} L_K(F)$$
as graded algebras
iff
$$K_0^{\Gamma}(L_K(E)) \cong K_0^{\Gamma}(L_K(F))$$
as pointed Γ-groups?

Classification

Let us look into "pointed" next...

Structure of K_0^Γ

- An abelian group
- with a Γ-action, and
- **a pre-order** $\leq$ (from $[P] \leq [Q]$ iff P is isomorphic to a summand of Q).

There is a special element $[R]$ in $K_0(R)$ which is an **order-unit** (because for every $a \in K_0(R)$, there is a positive integer n such that $-n[R] \leq a \leq n[R]$).

$K_0(R)$ considered with an order-unit u is said to be **pointed**.

"Being pointed" and "having Γ-action" matter

$$E = \begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd} \qquad F = \begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

$$L_K(E) \cong_{\mathrm{gr}} \mathbb{M}_3(K)(0,1,2) \quad \not\cong_{\mathrm{gr}} \quad L_K(F) \cong_{\mathrm{gr}} \mathbb{M}_3(K)(0,1,1).$$

$$(G_E^\Gamma, [1]) \cong (\mathbb{Z}[x, x^{-1}], 1 + x^{-1} + x^{-2}) \quad \not\cong$$
$$(G_F^\Gamma, [1]) \cong (\mathbb{Z}[x, x^{-1}], 1 + x^{-1} + x^{-1}).$$

Also

$$E = \begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[loop right] \end{tikzcd} \qquad F = \begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \arrow[l, bend left] \end{tikzcd}$$

$$L_K(E) \cong_{gr} \mathbb{M}_2(K[x, x^{-1}])(0,1) \not\cong_{\mathrm{gr}} L_K(F) \cong_{\mathrm{gr}} \mathbb{M}_2(K[x^2, x^{-2}])(0,1)$$

Γ acts trivially on $(G_E^\Gamma, [1])$ and non-trivially on $(G_F^\Gamma, [1])$.

Known classifications

Roozbeh (circa 2011) – the conjecture holds for finite **polycephaly** graphs (every path leads to a sink, a rose or a cycle with no exits).

Ara and Pardo (2014) – a weaker version of the conjecture holds for finite graphs without sources and sinks.

Eilers, Ruiz, Sims (2020) – the conjecture and its C^*-algebra version hold for countable "amplified" graphs.

Known classifications (continued)

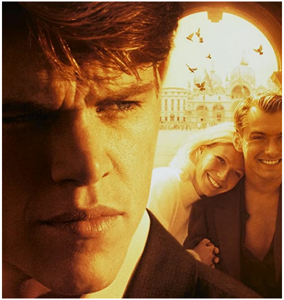
Roozbeh and me (circa 2016) – the involutive version of the conjecture holds for row-finite, no-exit graphs in which every infinite path ends in a sink or a cycle.

The proof relies on representing LPAs as (ultra)matricial algebras and using properties graded fields.

In 2020, Roozbeh and I were looking for...

... a graph-oriented (not matrix-oriented) approach.

So, we went back to the Talented Mr. Monoid and its structure.

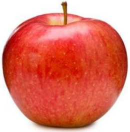


In particular, for $a \in M_E^\Gamma$, the relation $a < x^n a$ is not possible for any positive n. So, there are three possibilities:

1. $a = x^n a$ for some positive n. Such a is **periodic**.
2. $a > x^n a$ for some positive n. Such a is **aperiodic**.
3. a and $x^n a$ are not comparable for any positive n. Such a is **incomparable** (periodic or aperiodic = **comparable**).

Comparable and incomparable

Comparable



a $\quad$ $x^n a$

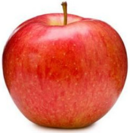
a is periodic

a $\quad$ $x^n a$

a is aperiodic

Incomparable

a



$x^n a$ for any n

a is incomparable

Examples

To understand examples a bit better: Look at the free Γ-monoid F_E^Γ generated by the vertices and look at a relation $\to$ "defined by the axioms" (i.e. $v \to \sum_{e \in \boldsymbol{s}^{-1}(v)} x\boldsymbol{r}(e)$ if v is regular). Then M_E^Γ is the equivalence closure of $\to$).

For example, for

$$\begin{tikzcd} \bullet^v \arrow[r] & \bullet^w \arrow[loop right] \end{tikzcd}$$

$$w \to xw \quad \Rightarrow \quad [w] = x[w] \quad \Rightarrow \quad [w] \text{ is } \textbf{periodic}.$$

$$v \to xw \quad \Rightarrow \quad [v] = x[w] = [w] \quad \Rightarrow \quad [v] \text{ is } \textbf{periodic} \text{ also.}$$

For

$$\begin{tikzcd} \bullet^v & \bullet^w \arrow[l] \arrow[loop right] \end{tikzcd}$$

$$w \to xw + xv \quad \Rightarrow \quad [w] > x[w] \quad \Rightarrow \quad [w] \text{ is } \textbf{aperiodic}.$$

$$v \to \text{nothing} \Rightarrow [v] \text{ is } \textbf{incomparable}.$$

Types of vertices

If v is a sink, $[v]$ is **incomparable**.

If v is an infinite emitter

$[v]$ is **aperiodic** if v is in a cycle.

$[v]$ is **incomparable** otherwise.

If v is in a cycle,

$[v]$ is **periodic** if the cycle has no exits,

$[v]$ is **aperiodic** otherwise.

$[a]$ is comparable iff $a \to b$ for some "stationary" element b of F_E^Γ.

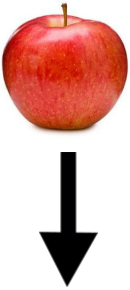
b is **stationary** iff all generators are either on cycles or on exits from cycles which contain other generators of b.

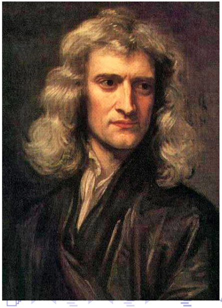
A taste of various characterizations

$[a] \in M_E^\Gamma$ is **comparable**	iff	$a \to b$ for some stationary b.
$[a] \in M_E^\Gamma$ is **periodic**	iff	$a \to b$ for some stationary b with no generators on cycles with exits.
Some $[a] \in M_E^\Gamma$ is **comparable**	iff	E has a cycle.
Some $[a] \in M_E^\Gamma$ is **periodic**	iff	E has a cycle without an exit
Every $[a] \in M_E^\Gamma$ is **periodic**	iff	E is row-finite, no-exit, without sinks, with infinite paths ending in cycles.

Some corollaries

1. M_E^Γ <u>**recognizes**</u> the following properties:
   - E being acyclic.
   - E having a cycle with/without an exit.
   - E being no-exit.
   - E being row-finite, no-exit, without sinks and with infinite paths ending in cycles.





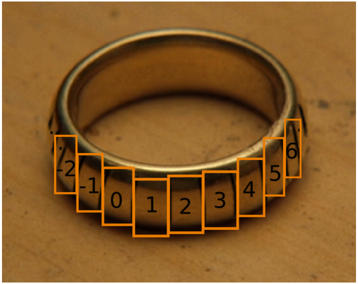
2. The main results of Roozbeh-Huanhuan paper on the Talented Monoid (J. Algebra, vol 547, 2020) hold **without the requirement that E is <u>row-finite</u>**.

Another corollary...

... related to the characterization of the **cross product LPAs and skew group ring LPAs** (Roozbeh-Lia, 2020)

Characterization of when $L_K(E)$ is strongly graded via M_E^Γ (equivalently of G_E^Γ).

graded ring



strongly graded ring

Idea for the future

The three classes match the polycephaly graphs scenario:

periodic	$\leftrightsquigarrow$	the comet part
aperiodic	$\leftrightsquigarrow$	the rose part
incomparable	$\leftrightsquigarrow$	the acyclic part

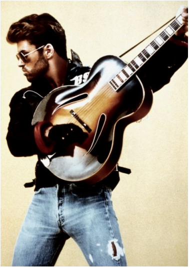
Not just that every element of M_E^Γ is periodic, aperiodic or incomparable, but it is a sum of a periodic, an aperiodic and an incomparable parts.

Such representation **exists**, but uniqueness is still a problem.

The strong version of the conjecture

K_0^Γ is a **full** and...

... **faith**ful functor.



The strong version, in fact, was shown for polycephaly and row-finite no-exit etc graphs mentioned before.

Relation with another conjecture

The **Isomorphism Conjecture.**

$L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$ **as rings** iff $C^*(E) \cong C^*(F)$ **as $*$-algebras.**

Formulated by Gene Abrams and Mark Tomforde. Note that

$L_{\mathbb{C}}(E) \cong L_{\mathbb{C}}(F)$ **as $*$-algebras** $\Rightarrow$ $C^*(E) \cong C^*(F)$ **as $*$-algebras.**

Mark

Gene

The graded (non-involutive) version

The **Graded Isomorphism Conjecture**.

$L_K(E) \cong L_K(F)$ **as graded rings** iff
$L_K(E) \cong L_K(F)$ **as graded algebras**.

Graded Classification $\Rightarrow$ **Graded Isomorphism**

References, slides: liavas.net