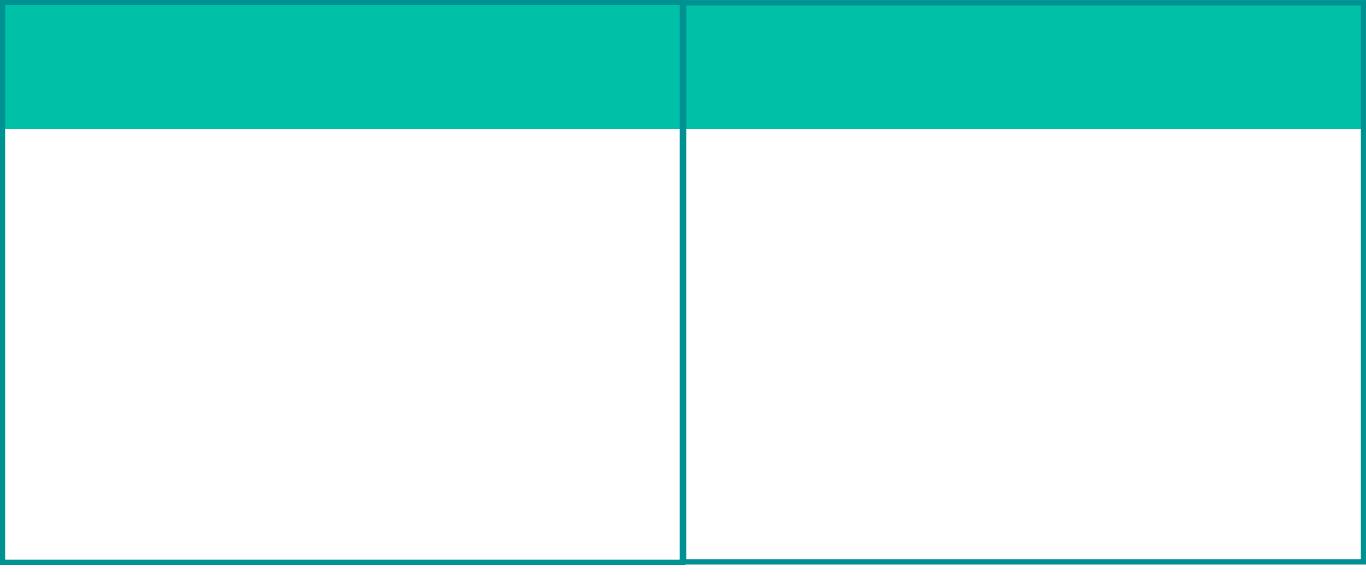
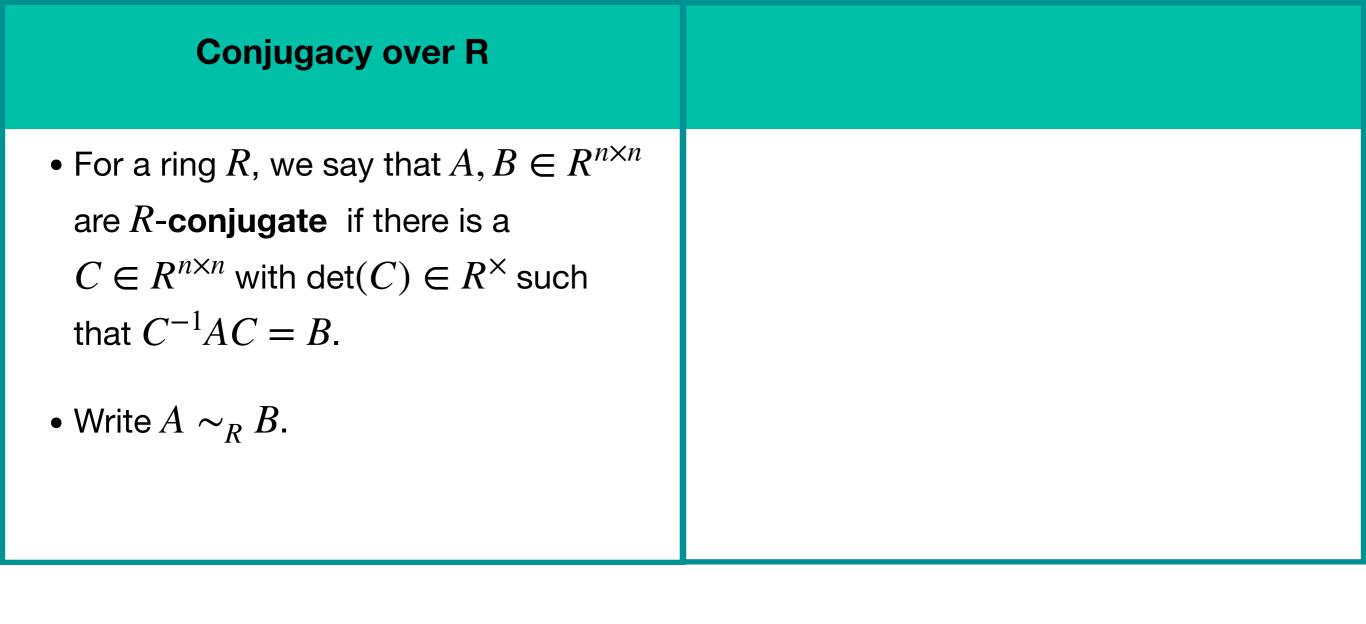
Conjugacy of Integral Matrices over Algebraic Extensions

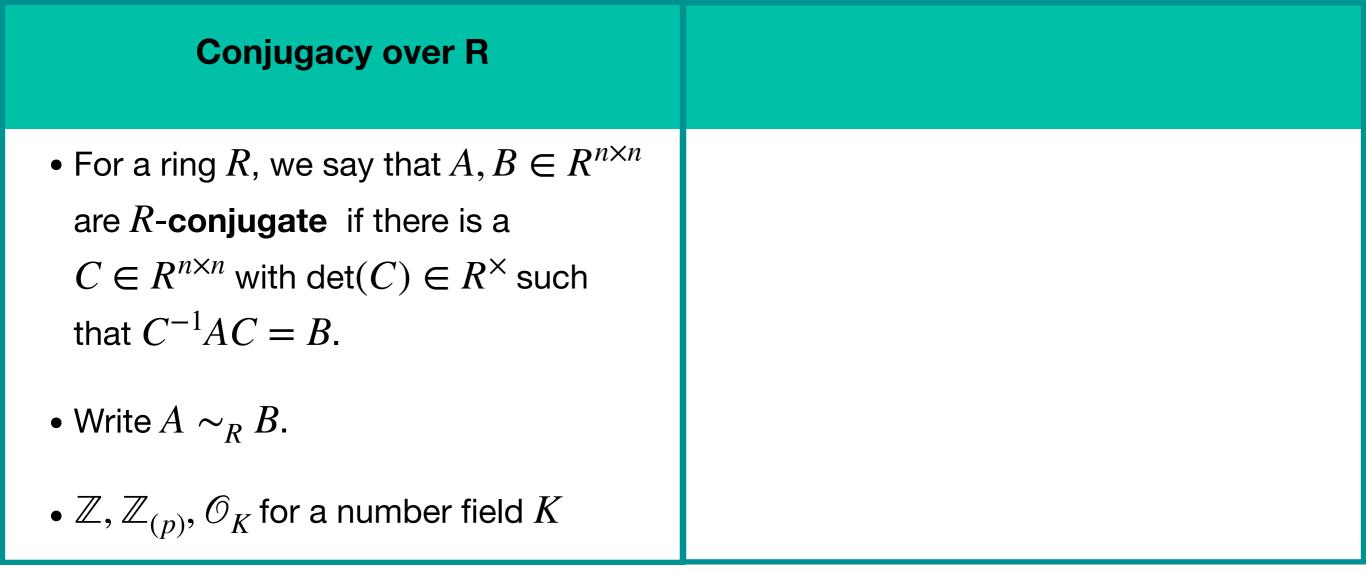
Rebecca Afandi

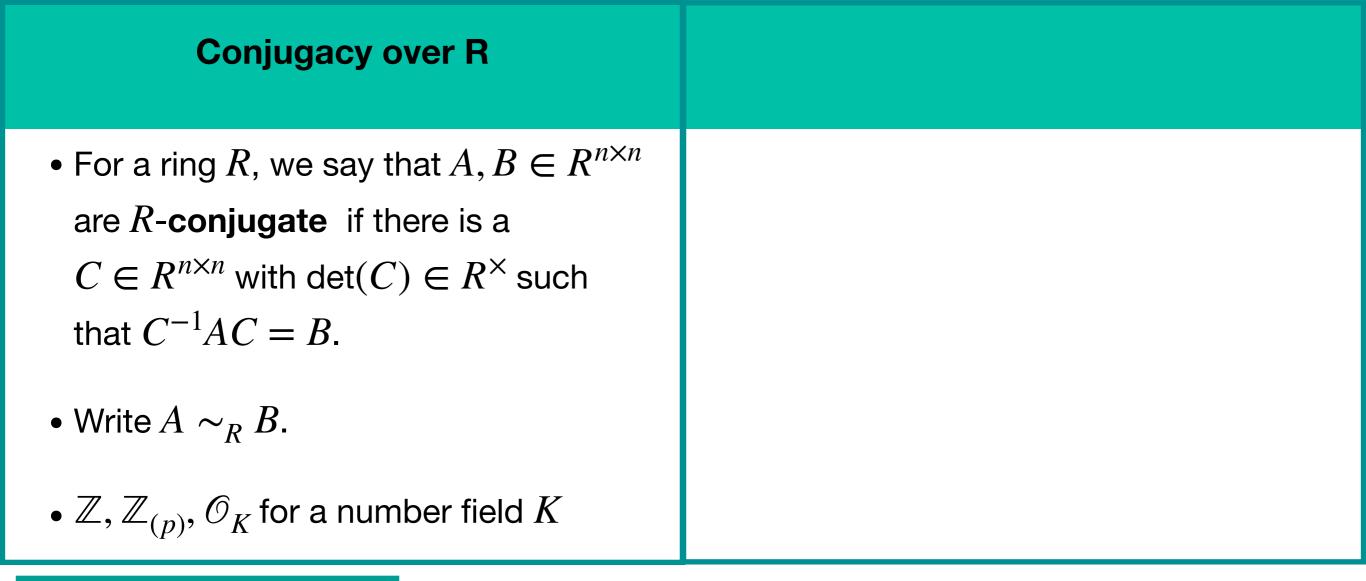


Conjugacy over R • For a ring R, we say that $A, B \in \mathbb{R}^{n \times n}$ are *R*-conjugate if there is a $C \in \mathbb{R}^{n \times n}$ with $det(C) \in \mathbb{R}^{\times}$ such that $C^{-1}AC = B$.

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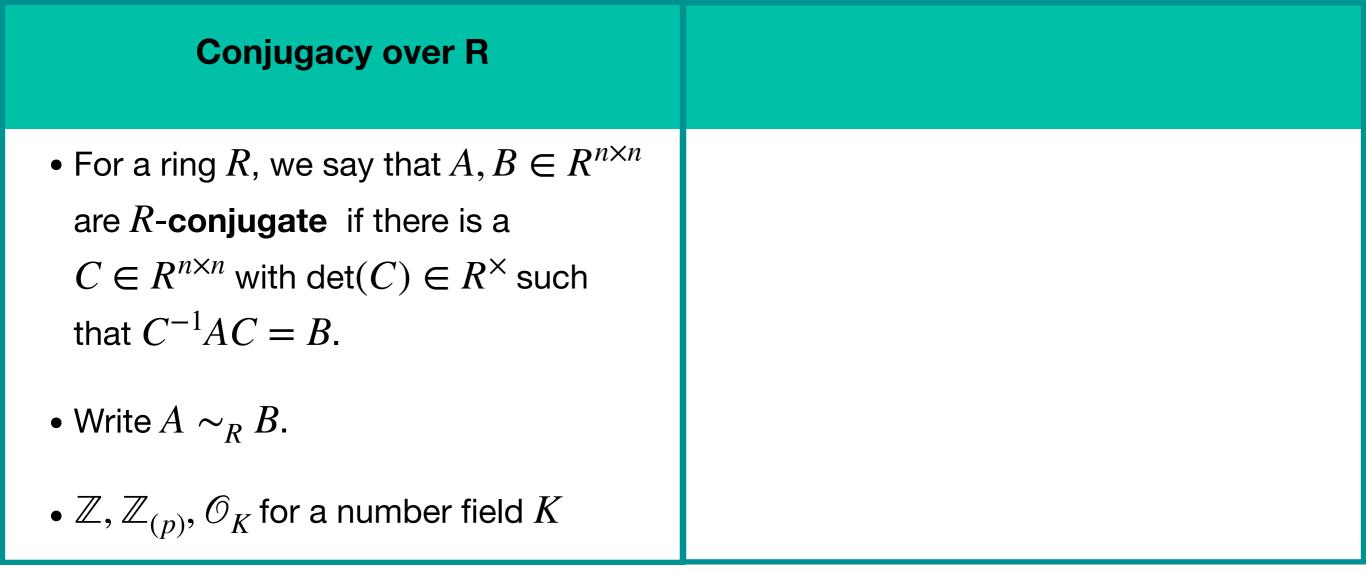




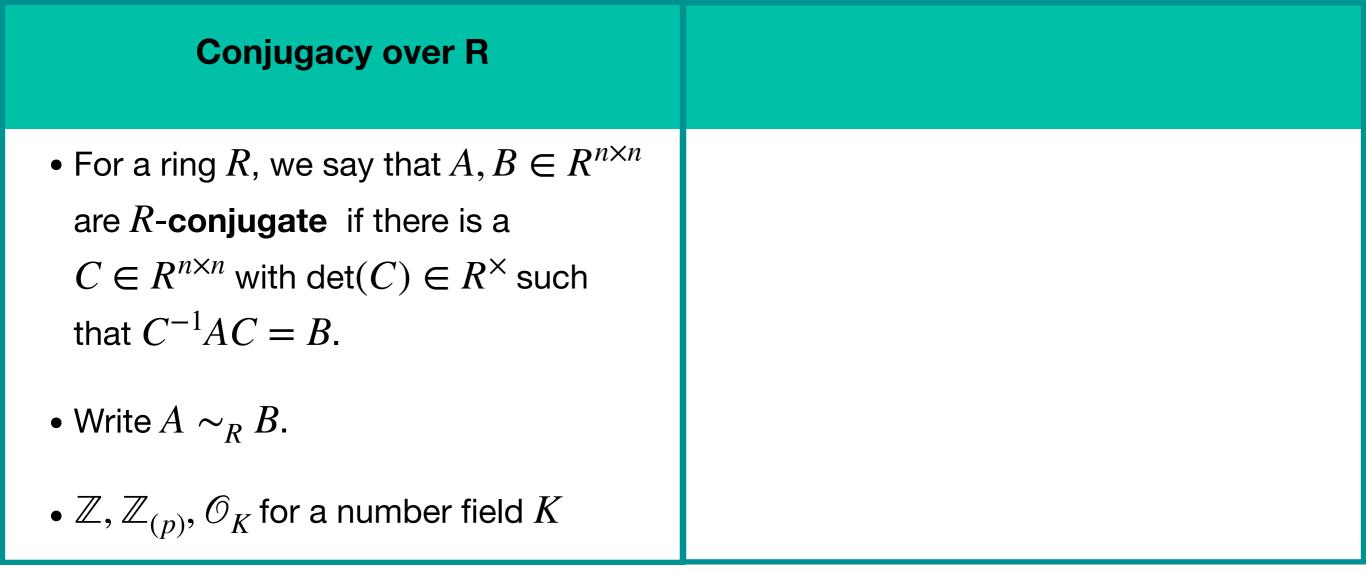


The localization of \mathbb{Z} at p is $\mathbb{Z}_{(p)} = \{\frac{a}{b} : a, b \in \mathbb{Z}, p \nmid b\}$

Rebecca Afandi



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 For a ring <i>R</i>, we say that are <i>R</i>-conjugate if the <i>C</i> ∈ <i>R^{n×n}</i> with det(<i>C</i>) that <i>C</i>⁻¹<i>AC</i> = <i>B</i>. Write <i>A</i> ~_{<i>R</i>} <i>B</i>. ℤ, ℤ_(p), 𝔅_K for a numb 	re is a $\in R^{\times}$ such			
	\mathcal{O}_{K} is the set of algebraic integelements (elements with monic integer minimal polynomial p	ral nents egral		



Conjugacy over R	R is a field
• For a ring R , we say that $A, B \in R^{n \times n}$ are R -conjugate if there is a $C \in R^{n \times n}$ with $det(C) \in R^{\times}$ such that $C^{-1}AC = B$.	
• Write $A \sim_R B$. • $\mathbb{Z}, \mathbb{Z}_{(p)}, \mathcal{O}_K$ for a number field K	

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• Write $A \sim_R B$. • $\mathbb{Z}, \mathbb{Z}_{(p)}, \mathcal{O}_K$ for a number field K	• $\mathcal{M}_f = \{A \in \mathbb{Z}^{n \times n} : \det(xI - A) = f\}$

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$$\mathbb{Z}$$
-conjugacy within \mathcal{M}_f for $f = x^2 + 5$

• Let $f = x^2 + 5$ and K be the number field $\mathbb{Q}(\alpha) \cong \mathbb{Q}[x]/(f)$. Note: $\mathcal{O}_K = \mathbb{Z}[\alpha]$. Some $\mathbb{Z}[\alpha]$ -fractional ideals in K are:

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- The fractional ideal classes form the ideal class group, denoted by $\operatorname{Pic}(\mathbb{Z}[\alpha])$. The class number is the order of the class group. $(h_K = 2 \text{ for } K = \mathbb{Q}(\alpha).)$

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$$\begin{array}{l} \alpha \cdot 1 = 0 \cdot 1 + 1 \cdot \alpha \\ \alpha \cdot \alpha = -5 \cdot 1 + 0 \cdot \alpha \end{array} \quad \text{so } \mathbb{Z}[\alpha] \text{ corresponds to } C_f = \begin{pmatrix} 0 & 1 \\ -5 & 0 \end{pmatrix}. \end{array}$$

 $\alpha \cdot 2 = -1 \cdot 2 + 2 \cdot (1 + \alpha)$ $\alpha \cdot (1 + \alpha) = -3 \cdot 2 + 1 \cdot (1 + \alpha)$ so *I* corresponds to $\begin{pmatrix} -1 & 2 \\ -3 & 1 \end{pmatrix}$.

Rebecca Afandi

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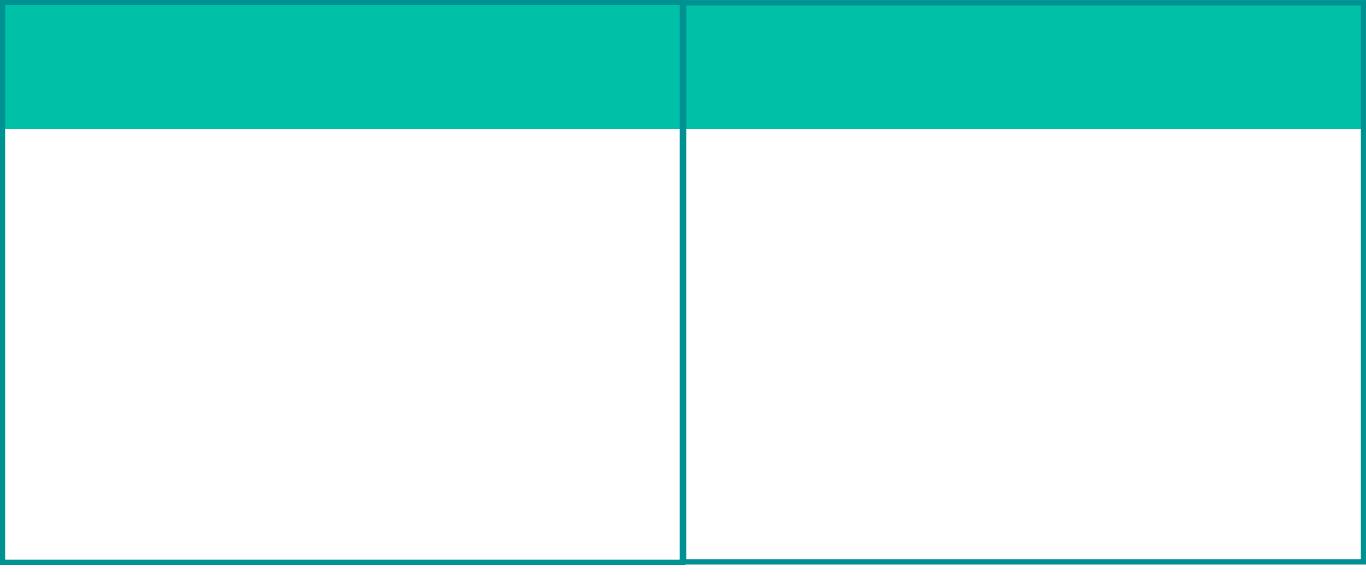
$$\mathbb{Z}[\alpha] \not\cong_{\mathbb{Z}[\alpha]} I \implies \begin{pmatrix} 0 & 1 \\ -5 & 0 \end{pmatrix} \not\sim_{\mathbb{Z}} \begin{pmatrix} -1 & 2 \\ -3 & 1 \end{pmatrix} \stackrel{) \cong \mathbb{Q}[x]/(f).}{|s \text{ in } K \text{ are:}}$$

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 $\alpha \cdot 2 = -1 \cdot 2 + 2 \cdot (1 + \alpha)$ so *I* corresponds to $\begin{pmatrix} -1 & 2 \\ -3 & 1 \end{pmatrix}$. $\alpha \cdot (1 + \alpha) = -3 \cdot 2 + 1 \cdot (1 + \alpha)$

Rebecca Afandi



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 For a ring <i>R</i>, we say that <i>A</i>, <i>B</i> ∈ <i>R^{n×n}</i> are <i>R</i>-conjugate if there is a <i>C</i> ∈ <i>R^{n×n}</i> with det(<i>C</i>) ∈ <i>R[×]</i> such that <i>C⁻¹AC</i> = <i>B</i>. Write <i>A</i> ~_{<i>R</i>} <i>B</i>. 	 All matrices with the same square-free characteristic polynomial are conjugate over a field. Let <i>f</i> ∈ Z[x] be monic and square-free of degree <i>n</i>. 	
• $\mathbb{Z}, \mathbb{Z}_{(p)}, \mathscr{O}_K$ for a number field K	• $\mathcal{M}_f = \{A \in \mathbb{Z}^{n \times n} : \det(xI - A) = f\}$	
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 f(x) irreducible with root α Let K = Q(α) 		

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• Write $A \sim_R B$. • $\mathbb{Z}, \mathbb{Z}_{(p)}, \mathcal{O}_K$ for a number field K	square-free of degree n . • $\mathcal{M}_f = \{A \in \mathbb{Z}^{n \times n} : \det(xI - A) = f\}$	
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• $f(x)$ irreducible with root α • Let $K = \mathbb{Q}(\alpha)$ • $\mathcal{M}_f/_{\sim_{\mathbb{Z}}} \leftrightarrow$ fractional $\mathbb{Z}[\alpha]$ -ideal classes in K	$f(x) = \prod_{i=1}^{m} f_i$ square-free with $\alpha = (\alpha_1, \dots, \alpha_m)$	

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that $C^{-1}AC = B$. • Write $A \sim_R B$.	• Let $f \in \mathbb{Z}[x]$ be monic and square-free of degree n .	
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• Let $K = \mathbb{Q}(\alpha)$	$f(x) = \prod_{i=1}^{m} f_i$ square-free with $\alpha = (\alpha_1, \dots, \alpha_m)$	
-ideal classes in K	$t K = \prod_{i=1}^{m} \mathbb{Q}(\alpha_i)$	
	$f'_{f'} \sim_{\mathbb{Z}} \leftrightarrow \text{ full } \mathbb{Z}[(\alpha_1, \dots, \alpha_m)] \text{-module classes}$	
Rebecca Afandi in	K Conjugacy of Integral Matrices	

Z-conjugacy within \mathcal{M}_{f} for $f = f_{1}f_{2}$ with $f_{1} = x^{2} + 4x + 7$, $f_{2} = x^{3} - 9x^{2} - 3x - 1$

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• Letting $K_i = \mathbb{Q}(\alpha_i) \cong \mathbb{Q}[x]/(f_i)$ we consider classes of $\mathbb{Z}[(\alpha_1, \alpha_2)]$ -modules within $K := K_1 \times K_2$.

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- $\mathcal{O}_K = \mathcal{O}_{K_1} \times \mathcal{O}_{K_2}$ but in general, fractional ideals are not products of fractional ideals in the $\mathscr{I}_{\mathbb{Z}[\alpha_i]}$.

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- $\mathcal{O}_K = \mathcal{O}_{K_1} \times \mathcal{O}_{K_2}$ but in general, fractional ideals are not products of fractional ideals in the $\mathscr{I}_{\mathbb{Z}[\alpha_i]}$.
- \mathcal{M}_{f_1} has 2 \mathbb{Z} -conjugacy classes and \mathcal{M}_{f_2} has 6 \mathbb{Z} -conjugacy classes, but \mathcal{M}_f has 852 \mathbb{Z} -classes.

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 $\varphi_{\mathbb{Z}}: \mathscr{I}_{\mathbb{Z}[\alpha]}/_{\cong \mathbb{Z}[\alpha]} \to \mathscr{M}_f/_{\sim \mathbb{Z}}$ $[I] \mapsto [A]$

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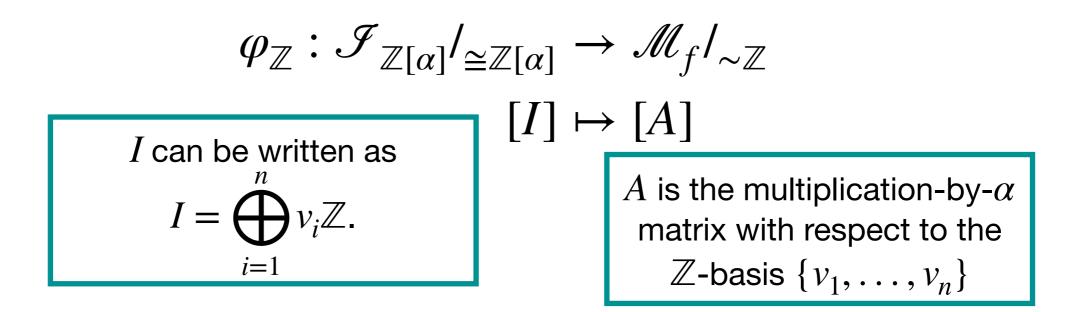
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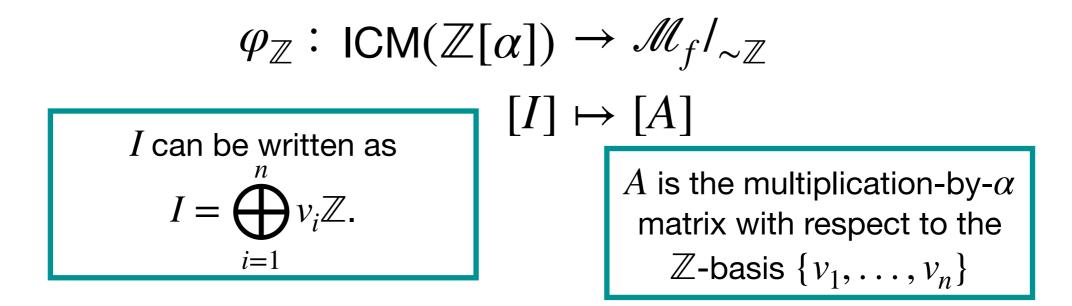
 $\mathscr{F}_{\mathbb{Z}[\alpha]}$ denotes the set of fractional $\mathbb{Z}[\alpha]$ -ideals.

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$$\begin{split} \varphi_{\mathbb{Z}} : \mathscr{I}_{\mathbb{Z}[\alpha]} /_{\cong \mathbb{Z}[\alpha]} \to \mathscr{M}_{f} /_{\sim \mathbb{Z}} \\ I \text{ can be written as} \\ I = \bigoplus_{i=1}^{n} v_{i} \mathbb{Z}. \end{split} \qquad \begin{bmatrix} I \end{bmatrix} \mapsto \begin{bmatrix} A \end{bmatrix} \end{split}$$

Rebecca Afandi





 $\varphi_{\mathbb{Z}}: \operatorname{ICM}(\mathbb{Z}[\alpha]) \to \mathscr{M}_f/_{\sim \mathbb{Z}}$ $[I] \mapsto [A]$

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• How to find $\psi_{\mathbb{Z}} := \varphi_{\mathbb{Z}}^{-1}$

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- How to find $\psi_{\mathbb{Z}} := \varphi_{\mathbb{Z}}^{-1}$
- For *f* irreducible, find $\overline{v} = (v_1, \dots, v_n)^t$ so that $A\overline{v} = \alpha\overline{v}$. Let $I = \bigoplus v_i \mathbb{Z}$ and let $\psi_{\mathbb{Z}}([A]) = [I]$.

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- How to find $\psi_{\mathbb{Z}} := \varphi_{\mathbb{Z}}^{-1}$
- For *f* irreducible, find $\overline{v} = (v_1, \dots, v_n)^t$ so that $A\overline{v} = \alpha \overline{v}$. Let $I = \bigoplus v_i \mathbb{Z}$ and let $\psi_{\mathbb{Z}}([A]) = [I]$.
- For f with m > 1 irreducible factors, let $A\overline{v}_i = \alpha_i \overline{v}_i$ and $\overline{v}_i = (v_{i1}, \dots, v_{in})^t$, then $\psi_{\mathbb{Z}}([A])$ has representative $I = (v_{11}, \dots, v_{m1})\mathbb{Z} \oplus \dots \oplus (v_{1n}, \dots, v_{mn})\mathbb{Z}$.

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• Letting
$$K := \mathbb{Q}(\alpha) = \mathbb{Q}[x]/(f)$$
, we have
 $\mathbb{Z}[\alpha] = 1\mathbb{Z} \oplus \alpha\mathbb{Z} \subsetneq \mathcal{O}_K = 1\mathbb{Z} \oplus \left(\frac{1+\alpha}{2}\right)\mathbb{Z}$

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- For a $\mathbb{Z}[\alpha]$ -ideal *I*, the **multiplicator ring** of *I* is (I : I).

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- For a $\mathbb{Z}[\alpha]$ -ideal I, the **multiplicator** ring of I is (I : I).

 $(I:J) = \{x \in \mathbb{Q}(\alpha) : xJ \subseteq I\}$

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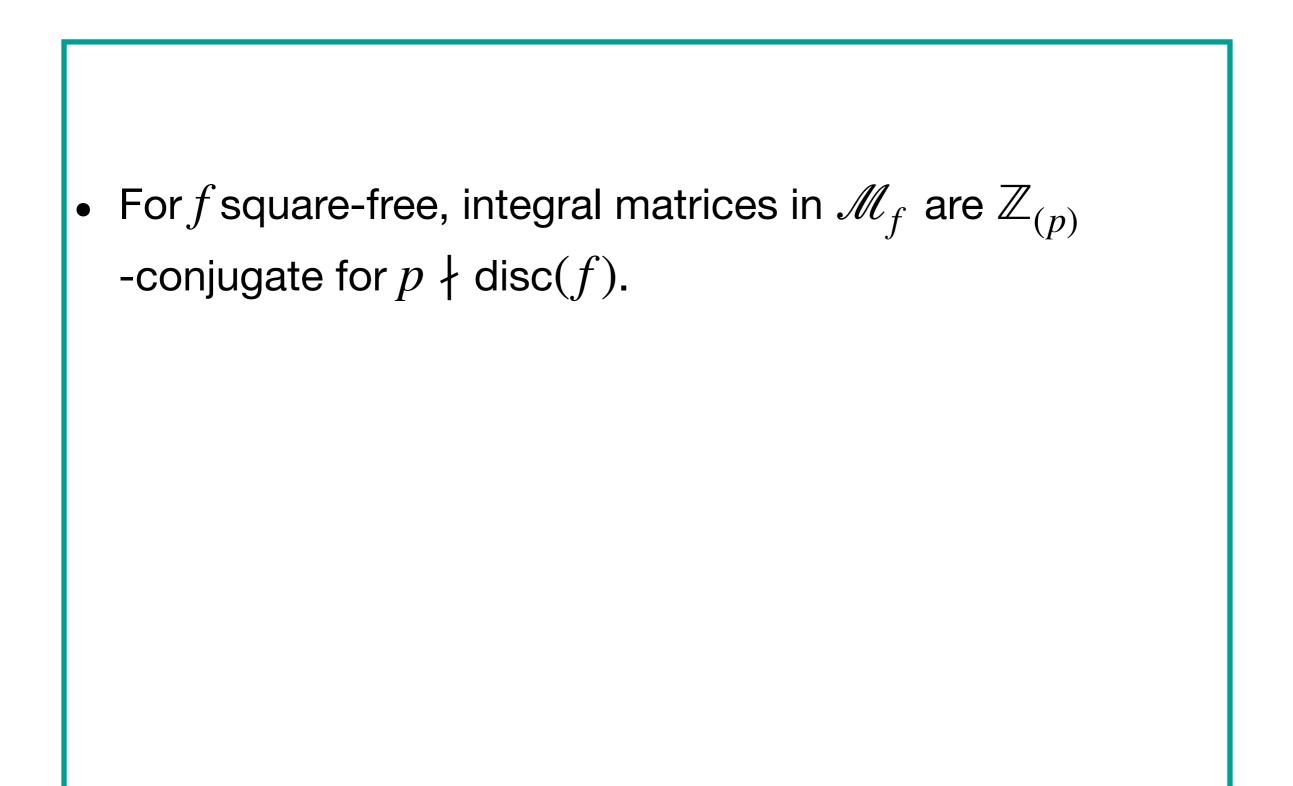
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Rebecca Afandi





 $\mathbb{Z}_{(p)}$ -conjugacy



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- I refer to matrices which satisfy $A \sim_{\mathbb{Z}_{(p)}} B$ for all primes p as **locally conjugate**.

$$A = \begin{pmatrix} 0 & -6 \\ 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 2 \\ -3 & 0 \end{pmatrix} \text{ have characteristic polynomial } c(x) = x^2 + 6, \text{ with } \text{disc}(c) = -24.$$

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Theorem of Guralnick (1980): $A \sim_{\mathbb{Z}_{(p)}} B$ over for all prime ideals $p \iff A \sim B$ over some finite integral extension E of \mathbb{Z} .

I refer to the problem of determining the algebraic extension over which locally conjugate matrices are conjugate as the **conjugacy extension problem.**

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Correspondence for an integral domain R

Rebecca Afandi

Correspondence for an integral domain *R*

• The Latimer and MacDuffee correspondence can be generalized to hold over any integral domain R.

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For
$$f = \prod_{i=1}^{m} f_i$$
, a **fractional** $R[\alpha]$ -ideal is an $R[\alpha]$ -module within
$$\prod_{i=1}^{m} \operatorname{Frac}(R)(\alpha_i)$$
 which is also a free R -module of rank deg (f) .

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$$\psi_{R} : \mathscr{M}_{f}/_{\sim R} \to \mathscr{I}_{R[\alpha]}/_{\cong_{R[\alpha]}}$$
$$[A]_{R} \mapsto [I]_{R[\alpha]}$$

There is a bijection

- The Latimer and MacDuffee correspondence can be generalized to hold over any integral domain R.
- Let $\mathscr{F}_{R[\alpha]}$ denote the set of fractional $R[\alpha]$ -ideals. $\psi_R : \mathscr{M}_f/_{\sim R} \to \mathscr{F}_{R[\alpha]}/_{\cong_{R[\alpha]}}$ There is a bijection $[A]_R \mapsto [I]_{R[\alpha]}$
- For $A \in \mathbb{Z}^{n \times n}$ and $\mathbb{Z} \subseteq R$, we have that $\psi_R([A]) = R \bigotimes_{\mathbb{Z}} \psi_{\mathbb{Z}}([A])$.

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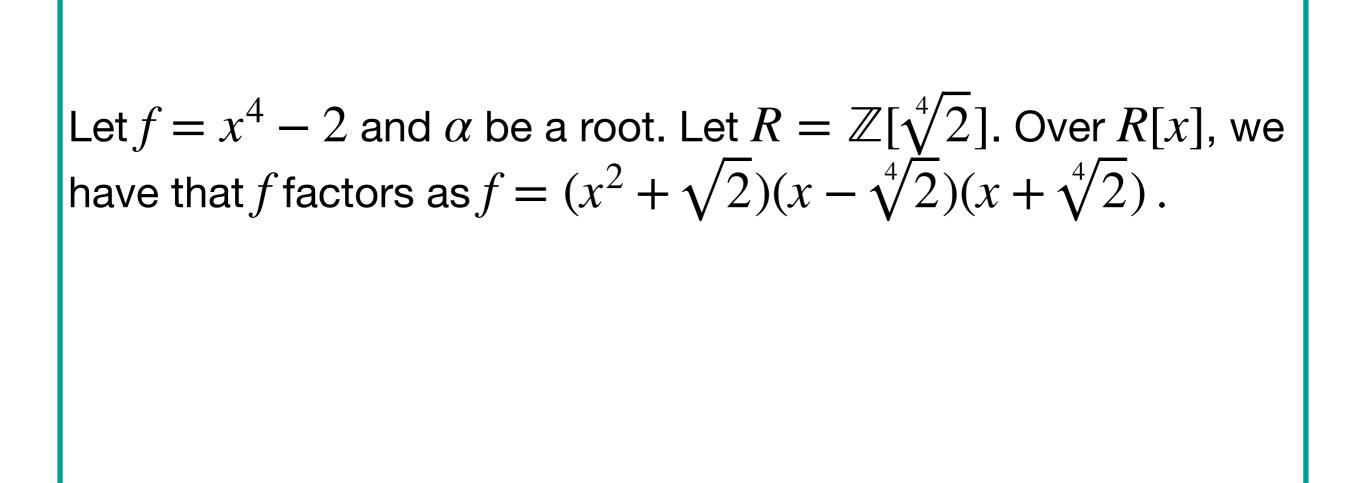
If
$$[A]_{\sim \mathbb{Z}} \leftrightarrow [I] = [\bigoplus p_i(\alpha)\mathbb{Z}],$$

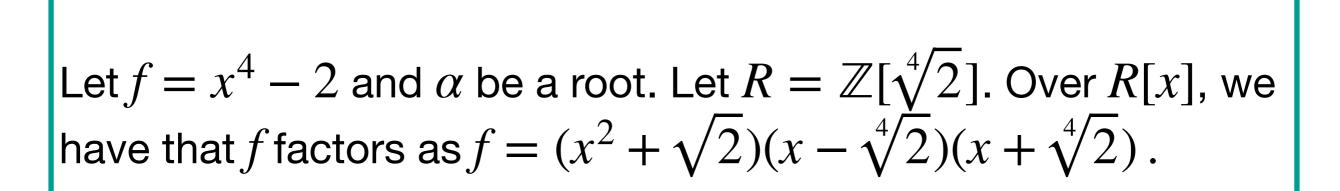
$$\begin{split} & [A]_{\sim R} \leftrightarrow [R \otimes I] = [\bigoplus p_i(\tilde{\alpha}) R] \text{ where the form of } \tilde{\alpha} \text{ depends on the} \\ & \text{factorization of } f \text{ in } R[x] \end{split}$$

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There is a bijection

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Let
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Let
$$f = x^4 - 2$$
 and α be a root. Let $R = \mathbb{Z}[\sqrt[4]{2}]$. Over $R[x]$, we have that f factors as $f = (x^2 + \sqrt{2})(x - \sqrt[4]{2})(x + \sqrt[4]{2})$.
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 $[C_f]_{\mathbb{Z}} \leftrightarrow [\mathbb{Z}[\alpha]]_{\mathbb{Z}[\alpha]} = [1\mathbb{Z} \oplus \alpha\mathbb{Z} \oplus \alpha^2\mathbb{Z} \oplus \alpha^3\mathbb{Z}]_{\mathbb{Z}[\alpha]}$ while $[C_f]_R \leftrightarrow [R \otimes_{\mathbb{Z}} \mathbb{Z}[\alpha]]_{R[\alpha]}$

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Input: Integral matrices *A* and *B* and a ring *R*. Tests if $A \sim_R B$ and if yes, returns $C \in GL_n(R)$ with $C^{-1}AC = B$.

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$$f = x^2 + 23$$
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$$A := \begin{pmatrix} -1 & 2 \\ -12 & 1 \end{pmatrix} \leftrightarrow R \otimes I := 2R \oplus (1+\alpha)R$$

and

$$B := \begin{pmatrix} 1 & 4 \\ -6 & -1 \end{pmatrix} \leftrightarrow R \otimes J := 4R \oplus (-1 + \alpha)R$$

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Note: *A* and *B* are locally conjugate iff $\mathbb{Z}_{(p)} \otimes I \cong_{\mathbb{Z}_{(p)}[\alpha]} \mathbb{Z}_{(p)} \otimes J$ iff (I:I) = (J:J).

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In \mathcal{O} , $R \otimes (I : J) = (\gamma)$.

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 $\ln \mathcal{O}, R \otimes (I:J) = (\gamma).$

Then $R \otimes I = \gamma(R \otimes J)$. So $R \otimes I$ has R-bases

 $\{2, 1 + \alpha\}$ and $\{4\gamma, \gamma(-1 + \alpha)\}$.

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For a particular \mathbb{Z} -basis $\{\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3\}$ of R, we find that

$$C = \begin{pmatrix} -\mathscr{B}_1 + \mathscr{B}_3 & -\mathscr{B}_1 - \mathscr{B}_2 \\ 2\mathscr{B}_1 + 3\mathscr{B}_2 + \mathscr{B}_3 & -2\mathscr{B}_1 + 2\mathscr{B}_3 \end{pmatrix}$$

has determinant in R^{\times} and conjugates A to B.

Implementation of Algorithm

Rebecca Afandi

• Implemented algorithm for $R = \mathcal{O}_L$ and for matrices in \mathcal{M}_f with f irreducible using subroutine IsPrincipal in Magma.

- Implemented algorithm for $R = \mathcal{O}_L$ and for matrices in \mathcal{M}_f with f irreducible using subroutine IsPrincipal in Magma.
- IsPrincipal is not valid for objects within a $\operatorname{Frac}(R)$ -algebra of the form $\prod_{i=1}^{m} \operatorname{Frac}(R)(\alpha_i)$

unless
$$R = \mathbb{Z}$$
 (or $m = 1$).

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Hilbert Class Fields

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Hilbert Class Fields

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- **Principal ideal theorem:** Let *L* denote the Hilbert class field of *K*. Every fractional \mathcal{O}_{K} -ideal is principal in \mathcal{O}_{L} .
- $\mathcal{M}_f = K := \mathbb{Q}(\alpha) = \mathbb{Q}[x]/(f) = L = HCF(K)$
- However, since $\alpha \in \mathcal{O}_L$, *f* factors further over $\mathcal{O}_L[x]$.

Rebecca Afandi

Let $f = x^2 + 5$ and $K = \mathbb{Q}(\alpha) \cong \mathbb{Q}[x]/(f)$.

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• Let *L* denote the Hilbert class field of *K* and $R = \mathcal{O}_L$. The *R*-conjugacy class of *A* corresponds to $R \otimes I = (2,2)R \oplus (\alpha + 1, -\alpha + 1)R$.

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- Let *L* denote the Hilbert class field of *K* and $R = \mathcal{O}_L$. The *R*-conjugacy class of *A* corresponds to $R \otimes I = (2,2)R \oplus (\alpha + 1, -\alpha + 1)R$.
- Letting $\{\mathscr{B}_1, \ldots, \mathscr{B}_4\}$ denote a \mathbb{Z} -basis for R, we have $2R \bigoplus (\alpha + 1)R = 2R \bigoplus (-\alpha + 1)R = (g)$ where $g = \mathscr{B}_1 2\mathscr{B}_1 \mathscr{B}_4$.

Rebecca Afandi

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Then $(2,2)R \oplus (\alpha + 1, -\alpha + 1)R$ is not principal and so $A \nsim_R C_f$ for R the ring of integers of the Hilbert class field of K.

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- (I:J) is principal in R

f	$\operatorname{disc}(f)$	h _K	A	$A \sim C_f$ over subfield of HCF?
$x^2 - x + 4$	-3 · 5	2	$\left(\begin{array}{rrr} -1 & 2 \\ -3 & 2 \end{array}\right)$	$x^2 + 2x + 4$
$x^{2} + 5$	$-2^{2} \cdot 5$	2	$\left(\begin{array}{cc} -1 & 2 \\ -3 & 1 \end{array}\right)$	No
$x^2 + 10$	$-2^{3} \cdot 5$	2	$\left \begin{array}{cc} 0 & 2 \\ -5 & 0 \end{array} \right)$	x ² + 2
$x^2 - x + 13$	-3 · 17	2	$\left \begin{array}{cc} -1 & 3 \\ -5 & 2 \end{array}\right)$	$x^2 + 8x + 19$
$x^2 + 13$	$-2^{2} \cdot 13$	2	$\left(\begin{array}{cc} -1 & 2 \\ -7 & 1 \end{array}\right)$	No
$x^2 - x + 6$	-23	3	$\left(\begin{array}{rrr} 0 & 2 \\ -3 & 1 \end{array}\right)$	$x^3 + 6x^2 + 9x - 23$
$x^2 - x + 8$	-31	3	$\left(\begin{array}{cc} -1 & 2 \\ -5 & 2 \end{array}\right)$	No
$x^2 + 17$	$-2^{2} \cdot 17$	4	$\left(\begin{array}{cc} -2 & 3 \\ -7 & 2 \end{array}\right)$	No
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$x^2 + 10$	$-2^3 \cdot 5$	2	$\begin{pmatrix} 0 & 2 \\ -5 & 0 \end{pmatrix}$ a	non-principal $\mathbb{Z}[\alpha]$ -ideal $^2+2$
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x ² + 5	$-2^{2} \cdot 5$	2	$A \sim_{R} C_{f} \text{ for } R \text{ the ring of } A$
x ² + 10	$-2^{3} \cdot 5$	2	$\int_{L} \frac{1}{2} \frac{1}{x^2 + 2} \frac{1}{x^2 + 2x + 4} = \frac{1}{2} \frac{1}{x^2 + 2x + 4}$
$x^2 - x + 13$	-3 · 17	2	$\begin{bmatrix} L - Q[x]/(x + 2x + 4) \\ -3 - 2 \end{bmatrix} - 8x + 19$
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•
$$C = \begin{pmatrix} -\mathscr{B}_2 & -1 - \mathscr{B}_4 \\ 3 + \mathscr{B}_2 + 3\mathscr{B}_4 & -1 - 2\mathscr{B}_2 - 2\mathscr{B}_3 - \mathscr{B}_4 \end{pmatrix}$$
 is a matrix in $\operatorname{GL}_2(\mathscr{O}_F)$ which conjugates \mathscr{C}_f to A .

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• Is there a way to implement the algorithm to test for $\operatorname{GL}_n(R)$ -conjugacy in the non-irreducible case? Need an algorithm that determines whether an ideal in $\prod_{m=1}^{m} \operatorname{Frac}(R)(\alpha_i) \text{ (as a Frac}(R)-\operatorname{algebra}) \text{ is principal.}$

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- How often does the method of searching through class fields succeed? Is there a nice classification for the cases in which the method works?
- Should we consider ray class fields which ramify at primes related to the discriminant of *f*?