Properties in A+B Rings

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April 6, 2022

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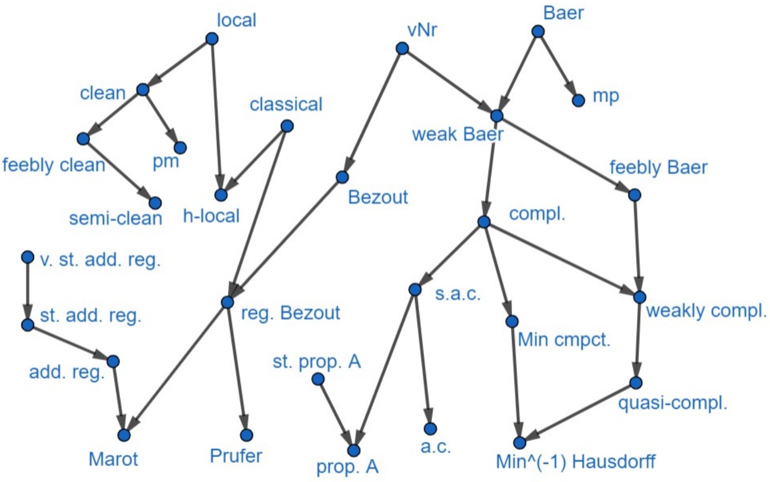
  - Ideals closed under taking double annihilators of elements/finite subsets
- $A + B$ rings used as examples
- Various authors investigated properties of $A + B$ rings, as well as various related constructions (sometimes still using the name '$A + B$ ring').
- Focus: look at a variety of properties and determine when $A + B$ satisfies each.

Some possible properties of reduced rings



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We let $Q(R)$ denote the *total quotient ring* of R (invert all non-zero divisors).

Constructing New Rings

Let A be a ring and $\mathcal{P} \subseteq \operatorname{Spec}(A)$ with index set $\mathcal{A}$ such that $\bigcap_{\alpha \in \mathcal{A}} P_\alpha = 0$.

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Take $a \in A$. When viewed in R, we have that a looks like:

$$(\ldots a + P_{i_1}, a + P_{(\alpha,1)}, a + P_{(\alpha,2)}, a + P_{(\alpha,3)}, \ldots, a + P_{i_2} \ldots)$$

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Note we are only changing $a \in R$ in finitely many components, based on the finitely many nonzero components of b.

Facts and details about $A + B$ rings

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A+B and A+QB Construction

Properties in A+B and A+QB Rings

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Spec$(A + B)$

$R = A + B$

Prime ideals of R which do not contain B

$\mathrm{Spec}(A + B)$

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Prime ideals of R which do not contain B

$i \in \mathcal{I}$, $M_i = \{r \in R \mid r(i) = 0\}$ are precisely the minimal primes not containing B.

$\mathrm{Spec}(A + B)$

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Prime ideals of R which do not contain B

$i \in \mathcal{I}$, $M_i = \{r \in R \mid r(i) = 0\}$ are precisely the minimal primes not containing B. All other primes not containing B are of the form $Q + M_i$ where $Q \in \mathrm{Spec}(A)$ contains P_i.

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Note: No proper ideal of R can contain more than one of the M_i's.

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Prime ideals of R containing B

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Note $R/B \cong A$. So the prime ideals of R containing B are in one-to-one correspondence with the prime ideals of A.

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$$P + B \text{ with } P \in \mathrm{Spec}(A)$$

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Identify A with its image in $\prod_{i \in \mathcal{I}} Q(A/P_i)$ and let $QB = \sum_{i \in \mathcal{I}} Q(A/P_i)$. $S = A + QB$ is our desired ring, with addition and multiplication defined coordinate-wise.

Note that (with the same base ring A and set of primes $\mathcal{P}$) $A + B \subseteq A + QB$; equality holds if and only if $\mathcal{P} \subseteq \mathrm{Max}(A)$.

Facts and details about $A + QB$ rings

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- $e_i S \cong Q(A/P_i)$

Spec$(A+QB)$

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Prime ideals of S which do not contain QB

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Spec$(A + QB)$

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Prime ideals of S containing QB

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$$P + QB \text{ with } P \in \mathrm{Spec}(A)$$

Special Elements in A, R, and S

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$R = A + B$, $S = A + QB$

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$R = A + B$, $S = A + QB$

- $r \in R$ (or S) is regular if and only if $r(i) \neq 0$ for all $i \in \mathcal{I}$.
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Lemma 1

Let $R = A + B$ and $S = A + QB$.

1. If an element $a \in A$ is $\mathcal{P}$-regular, then $a \in R$ (resp. S) is a regular element of R (resp. S).

Special Elements in A, R, and S

$R = A + B$, $S = A + QB$

- $r \in R$ (or S) is regular if and only if $r(i) \neq 0$ for all $i \in \mathcal{I}$.
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   - If $I + B \subseteq R$ (resp. S) is a regular ideal with $I \subseteq A$, then $I \subseteq A$ is a $\mathcal{P}$-regular ideal.

"Types" of Theorems

$R = A + B$, $S = A + QB$

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Examples

"Types" of Theorems

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Type 1 $\longrightarrow$ *von Neumann regular*

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Type 1 $\longrightarrow$ *von Neumann regular*

Type 2 $\longrightarrow$ *Marot*

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Type 1 $\longrightarrow$ *von Neumann regular*
Type 2 $\longrightarrow$ *Marot*
Type 3 $\longrightarrow$ *local*

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Type 4 $\longrightarrow$ *h-local* **

Relationship of property between A, R, and S

$R = A + B$, $S = A + QB$, [X] some property

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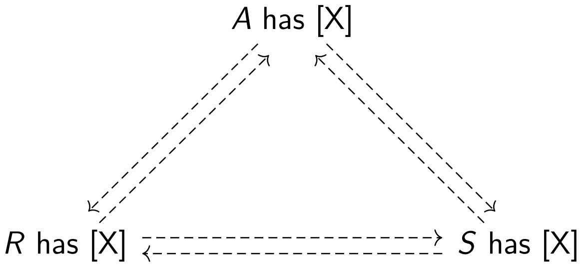

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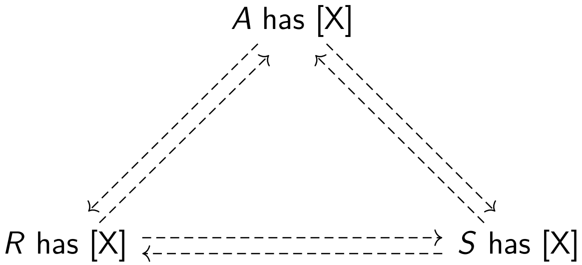

A goal: determine which implications hold and which do not

h-local

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3. For every $\mathcal{P}$-regular prime ideal $P \in \mathrm{Spec}(A)$, $P_\alpha \nsubseteq P$ for all $P_\alpha \in \mathcal{P}$.

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h-local (cont.)

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Let $S = A + QB$.

h-local (cont.)

Theorem 2

Let $S = A + QB$. S is h-local if and only if each $\mathcal{P}$-regular element of A has finite character and each $\mathcal{P}$-regular prime ideal of A is contained in a unique maximal ideal of A.

Note that this is just (1) and (2) from Theorem 1.

h-local (cont.)

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Note that this is just (1) and (2) from Theorem 1.

$M_i \in \mathrm{Max}(A + QB) \longrightarrow$ exclude (3)
$(A + QB)e_i \cong Q(A/P_i)$ a field (so h-local) $\longrightarrow$ exclude (4)

Proof of Theorem 1

$\Rightarrow$ (contrapositive)

Proof of Theorem 1

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1 or 2 fails:

Proof of Theorem 1

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① or ② fails: As the prime ideals of A correspond to the prime ideals of R containing B, it follows R is not h-local.

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Proof of Theorem 1

$\Rightarrow$ (contrapositive)

① or ② fails: As the prime ideals of A correspond to the prime ideals of R containing B, it follows R is not h-local.

③ fails: There exists $Q \subseteq \mathrm{Spec}(A)$ $\mathcal{P}$-regular properly containing some $P_\alpha \in \mathcal{P}$. Note $Q \subseteq N \in \mathrm{Max}(A)$. For $a \in Q$ $\mathcal{P}$-regular, $a \in R$ regular contained in $N + M_i$ with $i = (\alpha, n) \in \mathcal{I}$ for all $n \in \mathbb{N}$; R is not h-local.

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Proof of Theorem 1 (cont.)

$\Leftarrow$

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$\Leftarrow$

If $a + b \in R$ regular then (1) implies $a + b$ contained in only finitely many maximal ideals of R containing B.

Proof of Theorem 1 (cont.)

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$\Leftarrow$

If $a + b \in R$ regular then (1) implies $a + b$ contained in only finitely many maximal ideals of R containing B. (3) guarantees that $a \in R$ is not contained in any maximal ideals of R which do not contain B.

Proof of Theorem 1 (cont.)

$\Leftarrow$

If $a + b \in R$ regular then ❶ implies $a + b$ contained in only finitely many maximal ideals of R containing B. ❸ guarantees that $a \in R$ is not contained in any maximal ideals of R which do not contain B. ❹ implies that $a + b$ is contained in finitely many maximal ideals containing M_i for each $i \in \mathcal{I}$ such that $b(i) \neq 0$.

Proof of Theorem 1 (cont.)

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② implies that if $Q + B \in \mathrm{Spec}(R)$ is regular,

Proof of Theorem 1 (cont.)

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$\Leftarrow$

If $a + b \in R$ regular then 1 implies $a + b$ contained in only finitely many maximal ideals of R containing B. 3 guarantees that $a \in R$ is not contained in any maximal ideals of R which do not contain B. 4 implies that $a + b$ is contained in finitely many maximal ideals containing M_i for each $i \in \mathcal{I}$ such that $b(i) \neq 0$.

2 implies that if $Q + B \in \mathrm{Spec}(R)$ is regular, then it is contained in a unique maximal ideal of R, which has the form $N + B$ with $Q \subseteq N \in \mathrm{Max}(A)$.

Proof of Theorem 1 (cont.)

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If $a + b \in R$ regular then ❶ implies $a + b$ contained in only finitely many maximal ideals of R containing B. ❸ guarantees that $a \in R$ is not contained in any maximal ideals of R which do not contain B. ❹ implies that $a + b$ is contained in finitely many maximal ideals containing M_i for each $i \in \mathcal{I}$ such that $b(i) \neq 0$.

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Proof of Theorem 1 (cont.)

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If $a + b \in R$ regular then ① implies $a + b$ contained in only finitely many maximal ideals of R containing B. ③ guarantees that $a \in R$ is not contained in any maximal ideals of R which do not contain B. ④ implies that $a + b$ is contained in finitely many maximal ideals containing M_i for each $i \in \mathcal{I}$ such that $b(i) \neq 0$.

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④ implies that if $Q + M_i \in \mathrm{Spec}(R)$ is regular for some $i \in \mathcal{I}$, then it is contained in a unique maximal ideal of R, which has the form $N + M_i$ with $Q/P_i \subseteq N/P_i \in \mathrm{Max}(A/P_i)$.

□

Independence of conditions

1. Every $\mathcal{P}$-regular element of A has finite character in A.
2. Every $\mathcal{P}$-regular prime ideal of A is contained in a unique maximal ideal of A.
3. For every $\mathcal{P}$-regular prime ideal $P \in \mathrm{Spec}(A)$, $P_\alpha \nsubseteq P$ for all $P_\alpha \in \mathcal{P}$.
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We give examples to show that these conditions are independent of each other, and thus all are required for Theorem 1.
For each example, we give a ring A and set of primes $\mathcal{P} \subseteq \mathrm{Spec}(A)$ which will satisfy the stated conditions.

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Example 1: only (1) fails

Let $\beta\mathbb{N}$ be the Stone-Čech compactification of $\mathbb{N}$ and take $A = C(\beta\mathbb{N})$.

Independence of conditions

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Independence of conditions

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Example 2: only (2) fails

Independence of conditions

1. Every $\mathcal{P}$-regular element of A has finite character in A.
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3. For every $\mathcal{P}$-regular prime ideal $P \in \mathrm{Spec}(A)$, $P_\alpha \not\subseteq P$ for all $P_\alpha \in \mathcal{P}$.
4. A/P_α is h-local for all $P_\alpha \in \mathcal{P}$.

Example 2: only (2) fails

Let $D = T^{-1}K[X, Y]$ where
$T = K[X, Y] \setminus ((X, Y) \cup (X, Y - 1))$.

Independence of conditions

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Example 2: only (2) fails

Let $D = T^{-1}K[X, Y]$ where $T = K[X, Y] \setminus ((X, Y) \cup (X, Y - 1))$. Take $A = A' + QB'$ where $A' = D \times D$ and $\mathcal{P}' = \{D \times (0), (0) \times D\}$.

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Independence of conditions

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Example 3: only (3) fails

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3. For every $\mathcal{P}$-regular prime ideal $P \in \mathrm{Spec}(A)$, $P_\alpha \nsubseteq P$ for all $P_\alpha \in \mathcal{P}$.
4. A/P_α is h-local for all $P_\alpha \in \mathcal{P}$.

Example 3: only (3) fails

Let $A = K[X, Y]_{(X,Y)}$, the localization of $K[X, Y]$ at the maximal ideal (X, Y)

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4. A/P_α is h-local for all $P_\alpha \in \mathcal{P}$.

Example 3: only (3) fails

Let $A = K[X, Y]_{(X,Y)}$, the localization of $K[X, Y]$ at the maximal ideal (X, Y) and $\mathcal{P} = \{\text{nonzero principal prime ideals of } A\} \setminus (X)A$.

Independence of conditions

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Example 4: only (4) fails

Independence of conditions

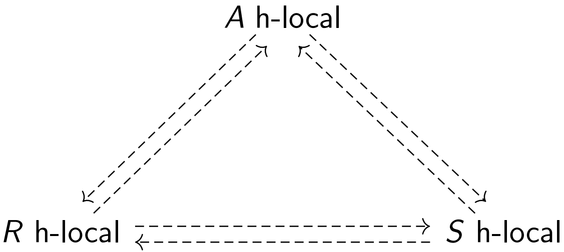
1. Every $\mathcal{P}$-regular element of A has finite character in A.
2. Every $\mathcal{P}$-regular prime ideal of A is contained in a unique maximal ideal of A.
3. For every $\mathcal{P}$-regular prime ideal $P \in \mathrm{Spec}(A)$, $P_\alpha \nsubseteq P$ for all $P_\alpha \in \mathcal{P}$.
4. A/P_α is h-local for all $P_\alpha \in \mathcal{P}$.

Example 4: only (4) fails

Let $A = K[X, Y, Z]$ and $\mathcal{P} = \{\text{nonzero principal prime ideals of } A\}$.

h-local between A, R, and S

$R = A + B$, $S = A + QB$



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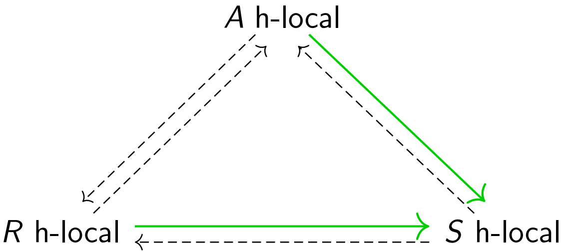

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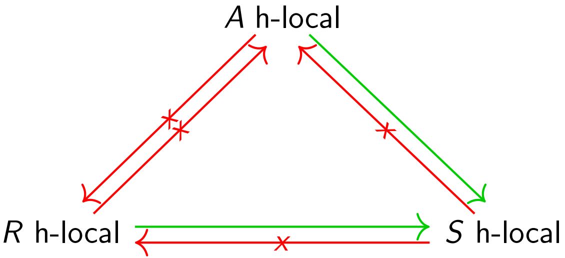

Thank you for listening!

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