# Tilting modules and tilting torsion pairs

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Let $A$ be an associative ring with $1 \neq 0$.

A left $A$-module ${}_AG$ is a progenerator if

- $G \leq^{\oplus} A^{(\alpha)}$ • $A \leq^{\oplus} G^{(\alpha)}$ • $\alpha \in \mathbb{N}$

A left $A$-module ${}_AT$ is a $n$-tilting module, $n \in \mathbb{N}$, if

- $0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_0 \longrightarrow T \longrightarrow 0$ $\quad P_i \leq^{\oplus} A^{(\alpha)}$
- $0 \longrightarrow A \longrightarrow T_0 \longrightarrow \cdots \longrightarrow T_n \longrightarrow 0$ $\quad T_i \leq^{\oplus} T^{(\alpha)}$
- $\mathrm{Ext}^i(T, T^{(\beta)}) = 0$ $\quad \forall i > 0,\ \forall$ cardinal $\beta$

${}_AT$ is a classical $n$-tilting module, if

- $\alpha \in \mathbb{N}$ (in such a case $\beta = 1$)

A progenerator is a classical 0-tilting module.

Let ${}_AT$ be a classical $n$-tilting module, and $B := \mathrm{End}({}_AT)$.

classical 0-tilting = progenerator

$$\mathrm{Hom}_A(T,?) : \quad A\text{-Mod} \quad \rightleftarrows \quad B\text{-Mod} \quad : T \otimes_B ?$$

Morita equivalences (1959)

classical $n$-tilting

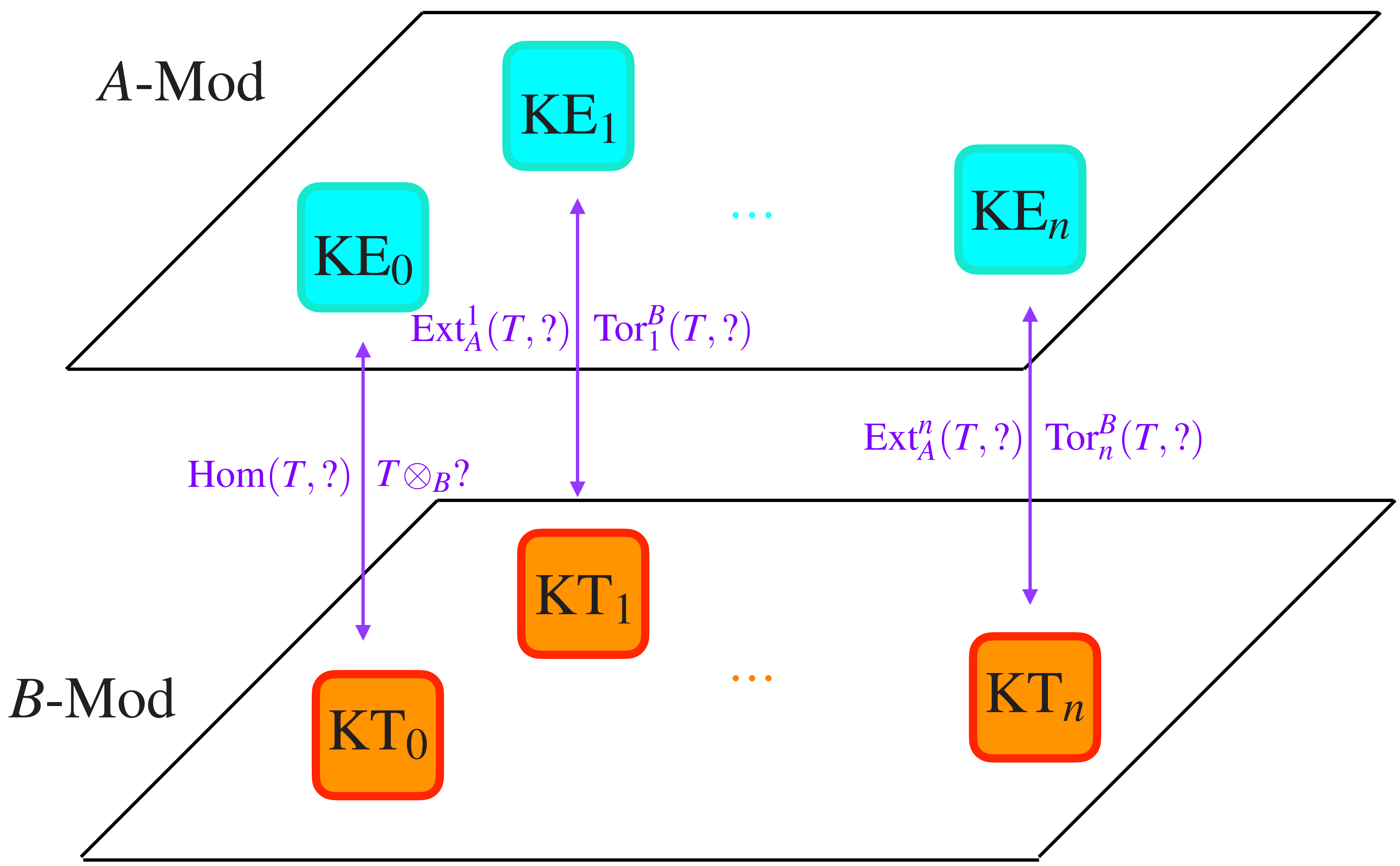
$$\mathrm{KE}_i := \{M \in A\text{-Mod} : \mathrm{Ext}_A^j(T,M) = 0 \ \forall j \neq i\}$$

$$\mathrm{Ext}^i(T,?) \downarrow \quad \uparrow \mathrm{Tor}_i(T,?)$$

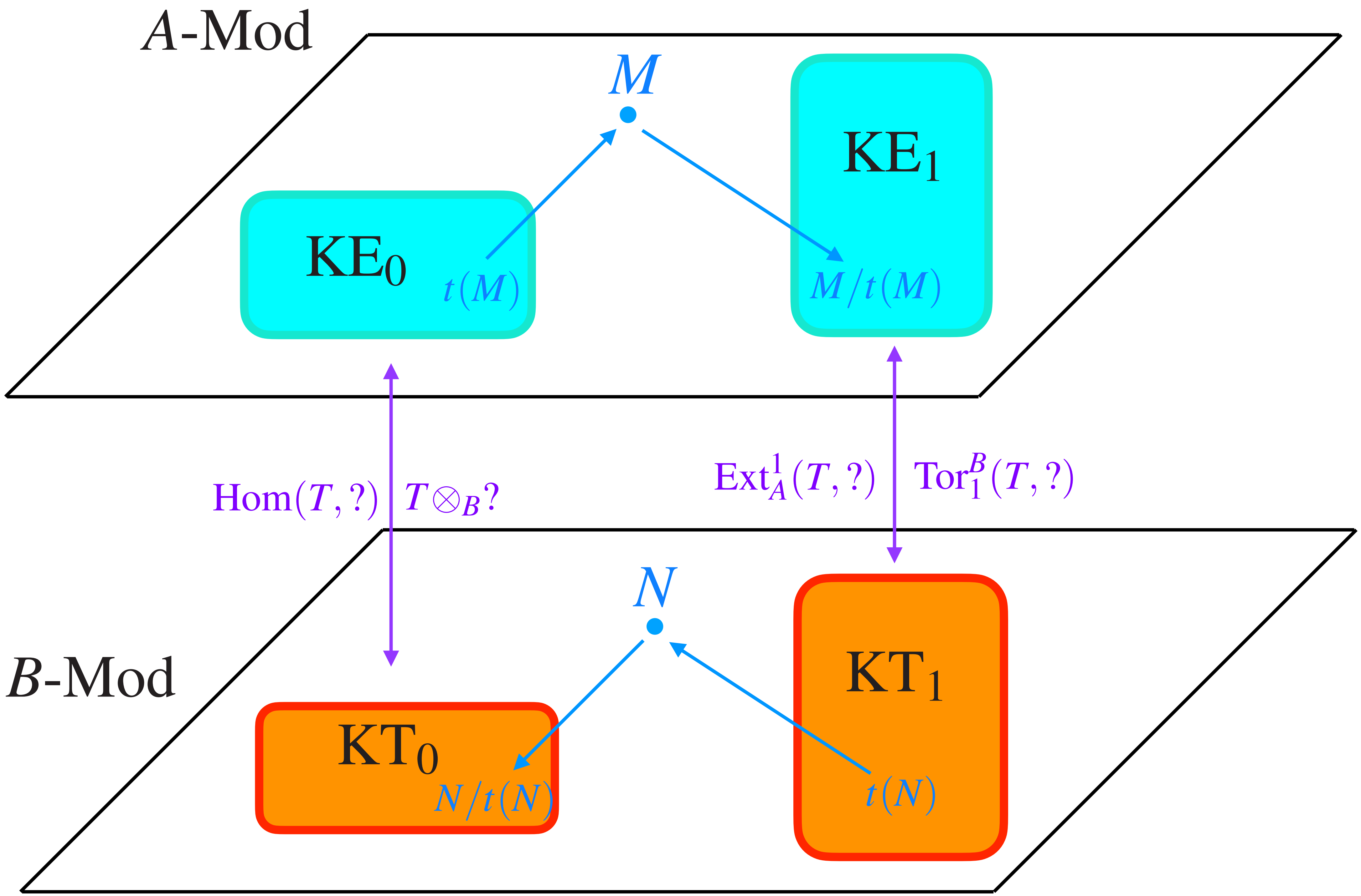
$$\mathrm{KT}_i := \{N \in B\text{-Mod} : \mathrm{Tor}_j^B(T,N) = 0 \ \forall j \neq i\}$$

Miyashita equivalences (1986)

$n \geq 1$ $\quad {}_A T$ classical $n$-tilting



## $n = 1$ $\quad {}_AT$ classical 1-tilting




$(\mathrm{KE}_0, \mathrm{KE}_1)$ $\quad$ are torsion pairs in $\quad$ $A$-Mod

$(\mathrm{KT}_1, \mathrm{KT}_0)$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $B$-Mod

$n = 1$ $\quad {}_A T$ classical 1-tilting

$$A\text{-Mod} \ni M = \begin{array}{|c|}\hline M/t(M) \\ \hline t(M) \\ \hline\end{array} \quad \begin{array}{l} \in \mathrm{KE}_1 \\ \in \mathrm{KE}_0 \end{array}$$

$$B\text{-Mod} \ni N = \begin{array}{|c|}\hline N/t(N) \\ \hline t(N) \\ \hline\end{array} \quad \begin{array}{l} \in \mathrm{KT}_0 \\ \in \mathrm{KT}_1 \end{array}$$

$n > 1$ $\quad {}_AT$ $\quad$ classical $n$-tilting

# Theorem [T - '02]

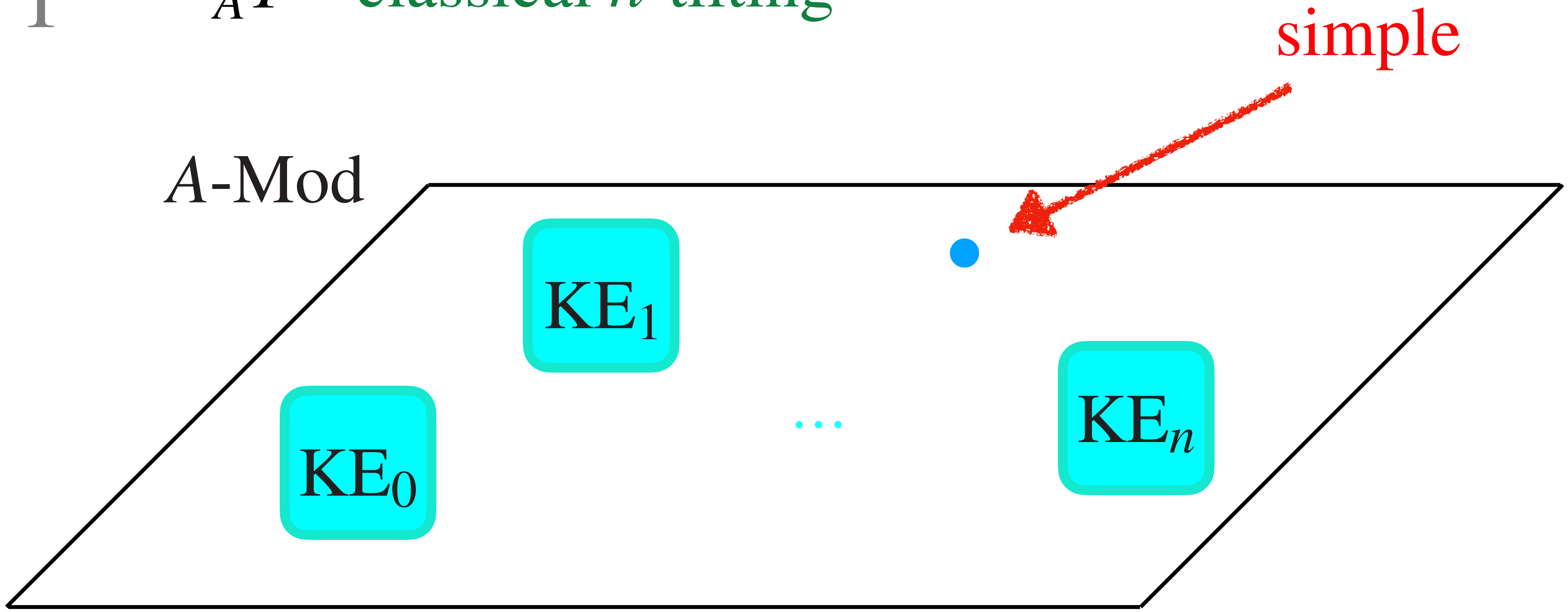
$$A\text{-Mod} \ni M = \begin{array}{ccc} \square & \in & \mathrm{KE}_n \\ \vdots & & \vdots \\ \square & \in & \mathrm{KE}_1 \\ \square & \in & \mathrm{KE}_0 \end{array}$$

iff $\mathrm{Tor}_i^B(T, \mathrm{Ext}_A^j(T, M)) = 0 \quad \forall i \neq j$

$$B\text{-Mod} \ni N = \begin{array}{ccc} \square & \in & \mathrm{KT}_0 \\ \square & \in & \mathrm{KT}_1 \\ \vdots & & \vdots \\ \square & \in & \mathrm{KT}_n \end{array}$$

iff $\mathrm{Ext}_A^i(T, \mathrm{Tor}_j^B(T, M)) = 0 \quad \forall i \neq j$

$n > 1$ $\quad {}_AT$ classical $n$-tilting



$k$ algebraically closed field

$A$ $k$-algebra $\quad 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \quad b \circ a = 0$

Indecomposable projectives: $\begin{smallmatrix}1\\2\end{smallmatrix}$ $\begin{smallmatrix}2\\3\end{smallmatrix}$ $3$

Indecomposable injectives: $1$ $\begin{smallmatrix}1\\2\end{smallmatrix}$ $\begin{smallmatrix}2\\3\end{smallmatrix}$

$${}_AT = 1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix}$$ is a classical 2-tilting

$$0 \to 3 \oplus 0 \oplus 0 \to \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 0 \oplus 0 \to \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix} \to {}_AT \to 0$$

Applying Hom(?, 2):

$$0 \leftarrow \mathrm{Hom}(\begin{smallmatrix}2\\3\end{smallmatrix}, 2) \xleftarrow{0} \mathrm{Hom}(\begin{smallmatrix}2\\3\end{smallmatrix}, 2)$$

$$\mathrm{Hom}(T,2) \cong \mathrm{Ext}(T,2) \cong \mathrm{Hom}(\begin{smallmatrix}2\\3\end{smallmatrix}, 2) \neq 0$$

What to do?

- Enlarge the classes $KE_i$:

  Jensen, Madsen and Su [2013]

  Lo [2013]

- Change the point of view:

  Fiorot, Mattiello, T [2016]

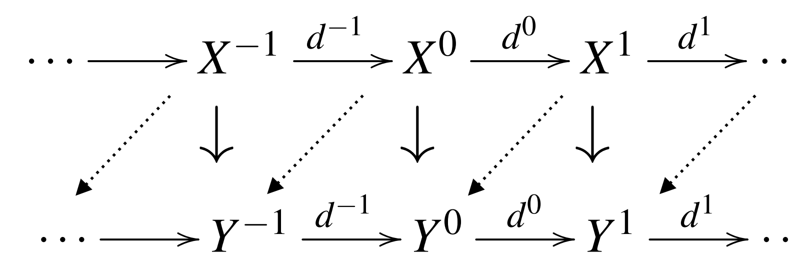
  Mattiello, Pavon, T [2020]

# The derived category $\mathscr{D}(A)$ of $A$-Mod

-2 -1 0 1

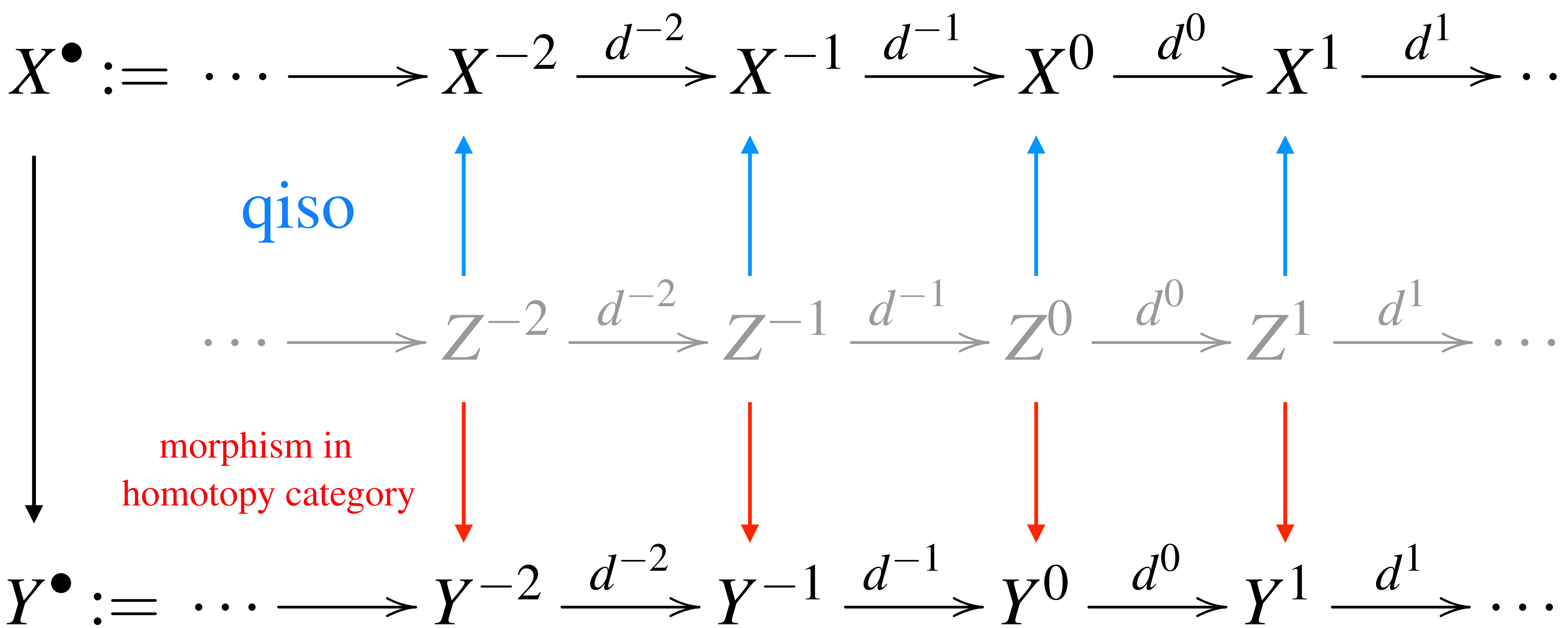
**Objects:** $X^\bullet := \cdots \longrightarrow X^{-2} \xrightarrow{d^{-2}} X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots$

**Morphisms:** chain complex morphisms,
modulo null homotopic ones,
formally inverting qiso



-2 -1 0 1

**Shift:** $X^\bullet[1] := \cdots \longrightarrow X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} \cdots$



X• := ··· → X^−2 →(d^−2) X^−1 →(d^−1) X^0 →(d^0) X^1 →(d^1) ···
qiso
··· → Z^−2 →(d^−2) Z^−1 →(d^−1) Z^0 →(d^0) Z^1 →(d^1) ···
morphism in
homotopy category
Y• := ··· → Y^−2 →(d^−2) Y^−1 →(d^−1) Y^0 →(d^0) Y^1 →(d^1) ···

$$X^\bullet := \cdots \longrightarrow X^{-2} \xrightarrow{d^{-2}} X^{-1} \xrightarrow{d^{-1}} X^{0} \xrightarrow{d^{0}} X^{1} \xrightarrow{d^{1}} \cdots$$

$$\Big\downarrow \quad \text{morphism in homotopy category} \qquad \Big\downarrow \qquad \Big\downarrow \qquad \Big\downarrow \qquad \Big\downarrow$$

$$I^\bullet := 0 \to \cdots \to I^{-2} \longrightarrow I^{-1} \xrightarrow{d^{-1}} I^{0} \xrightarrow{d^{0}} I^{1} \xrightarrow{d^{1}} \cdots$$

$\mathscr{D}(A)$ is NOT abelian

$\mathscr{D}(A)$ is triangulated

$$X^\bullet \xrightarrow{f} Y^\bullet \to Z^\bullet \to X^\bullet[1] \to Y^\bullet[1] \to \cdots$$

$$Y^\bullet \to Z^\bullet \to X^\bullet[1] \to Y^\bullet[1]$$

If $0 \to X \to Y \to Z \to 0$ is a short exact sequence in $A$-Mod, then

$$X \to Y \to Z \to X[1]$$

is a triangle in $\mathscr{D}(A)$

$X, Y$ in $A$-Mod

$$0 \to Y \to I_0 \to I_1 \to \cdots = 0 \to Y \to I(Y)$$

$$\begin{array}{ccccccc} 0 & \to & Y & \to & 0 & \to & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & I_0 & \to & I_1 & \to & \cdots \end{array}$$

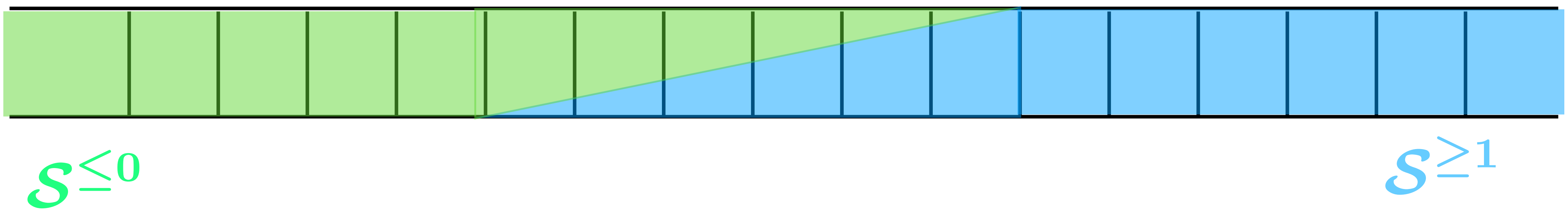
$$\mathrm{Hom}_{\mathscr{D}(A)}(X, Y[n]) \cong \mathrm{Hom}_{\mathscr{D}(A)}(X, I(Y)[n]) \cong \mathrm{Hom}_{\mathscr{K}(A)}(X, I(Y)[n])$$

$$\begin{array}{ccccccccccccc} & & & & 0 & \to & 0 & \to & \overset{0}{X} & \to & 0 & \to & \\ & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ I_0 & \to & I_1 & \to & \cdots & \to & I_{n-1} & \to & I_n & \to & I_{n+1} & \to & \cdots \end{array}$$

$$\mathrm{Hom}_A(X, I_{n-1}) \to \mathrm{Hom}_A(X, I_n) \to \mathrm{Hom}_A(X, I_{n+1})$$

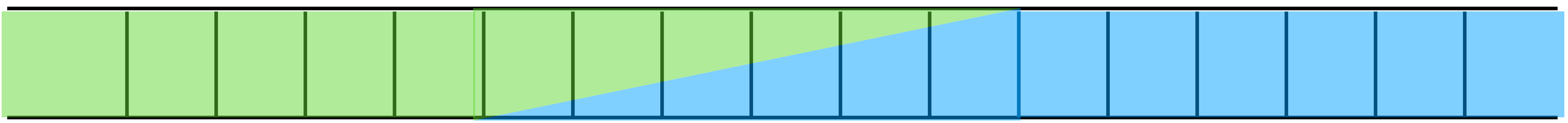
$$\boxed{\mathrm{Hom}_{\mathscr{D}(A)}(X, Y[n]) \cong \mathrm{Ext}_A^n(X, Y)}$$

**$t$-structure** pair $(\mathcal{S}^{\leq 0}, \mathcal{S}^{\geq 1})$ of full subcategories of $\mathscr{D}(A)$

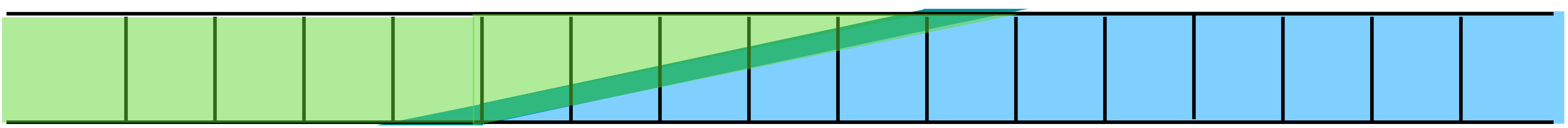



- $\mathcal{S}^{\leq 0}[1] =: \mathcal{S}^{\leq -1} \subseteq \mathcal{S}^{\leq 0}$ $\quad$ $\mathcal{S}^{\geq 1}[-1] =: \mathcal{S}^{\geq 2} \subseteq \mathcal{S}^{\geq 1}$
- $\mathrm{Hom}\,(\mathcal{S}^{\leq 0}, \mathcal{S}^{\geq 1}) = 0$
- $X^\bullet_{\leq 0} \to X^\bullet \to X^\bullet_{\geq 1} \to X^\bullet_{\leq 0}[1]$ $\quad \forall \quad X^\bullet \in \mathscr{D}(A)$

# heart of a $t$-structure



$\mathcal{S}^{\leq 0}$ $\mathcal{S}^{\geq 1}$



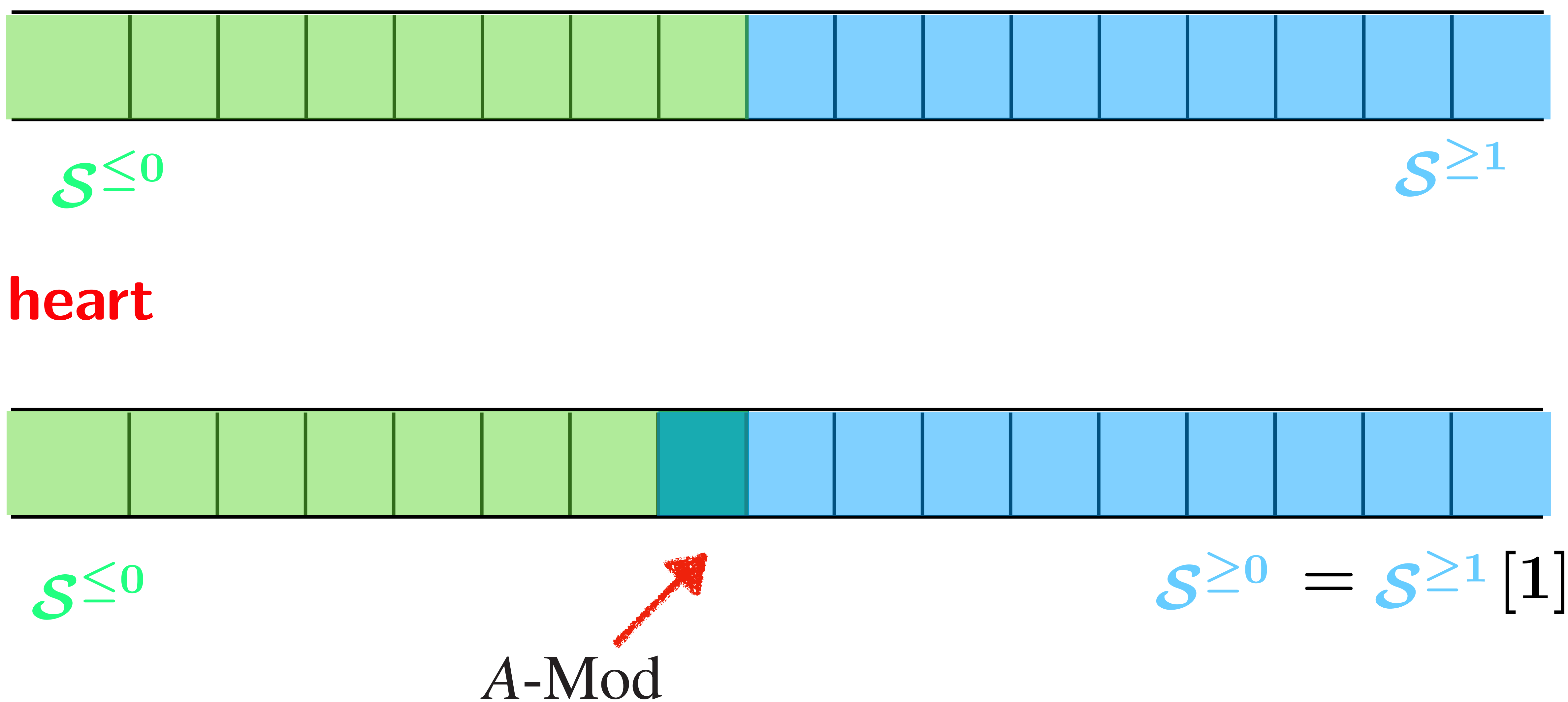
$\mathcal{S}^{\leq 0}$ $\mathcal{S}^{\geq 0} = \mathcal{S}^{\geq 1}[1]$

$= \mathcal{S}^{\leq 0} \cap \mathcal{S}^{\geq 0} =$ **heart**

**The heart of a $t$-structure is an abelian category !**

Beilinson, Bernstein, Deligne (1981)

# Natural $t$-structure



# Tilting $t$-structure $\mathscr{T}$

$$\mathcal{T}_{\leq 0} = \{X^\bullet \in \mathscr{D}(A) : \operatorname{Hom}_{\mathscr{D}(A)}(T, X^\bullet[i]) = 0 \text{ for every } i > 0\}$$

$$\mathcal{T}_{\geq 1} = \{X^\bullet \in \mathscr{D}(A) : \operatorname{Hom}_{\mathscr{D}(A)}(T, X^\bullet[i]) = 0 \text{ for every } i \leq 0\}$$

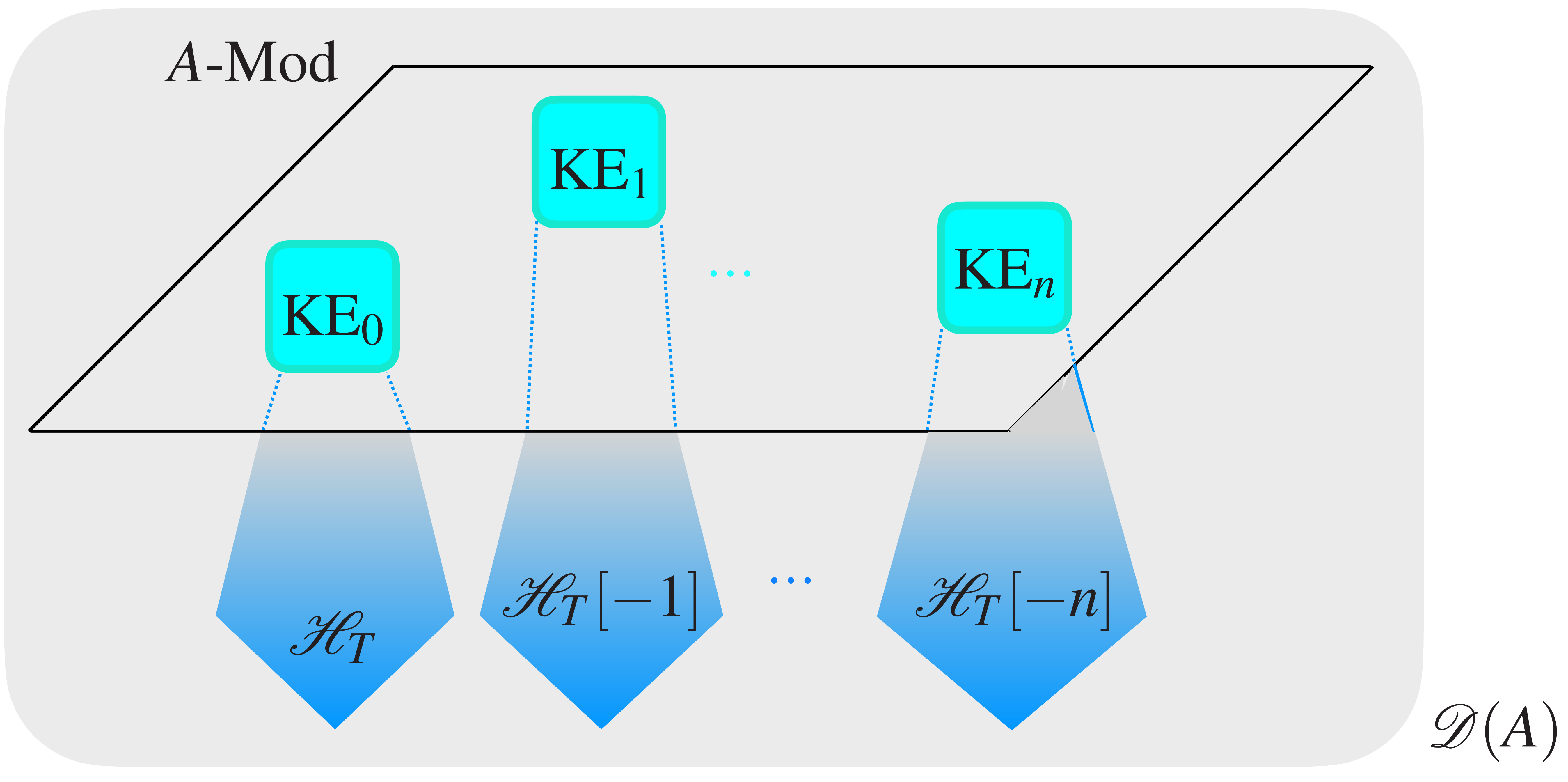
$$\mathcal{T}_{\geq 1}[1] = \mathcal{T}_{\geq 0} = \{X^\bullet \in \mathscr{D}(A) : \operatorname{Hom}_{\mathscr{D}(A)}(T, X^\bullet[i]) = 0 \text{ for every } i < 0\}$$

$$\mathscr{H}_{\mathscr{T}} = \mathcal{T}_{\leq 0} \cap \mathcal{T}_{\geq 0} = \{X^\bullet \in \mathscr{D}(A) : \operatorname{Hom}_{\mathscr{D}(A)}(T, X^\bullet[i]) = 0 \ \forall\, i \neq 0\}$$

$$\mathscr{H}_{\mathscr{T}}[-1] = \{X^\bullet[-1] \in \mathscr{D}(A) : \operatorname{Hom}_{\mathscr{D}(A)}(T, X^\bullet[i]) = 0 \ \forall\, i \neq 0\}$$

$$= \{X^\bullet \in \mathscr{D}(A) : \operatorname{Hom}_{\mathscr{D}(A)}(T, X^\bullet[i]) = 0 \ \forall\, i \neq 1\}$$

$$KE_e = A\text{-Mod} \cap \left(\mathscr{H}_{\mathscr{T}}[-e]\right)$$



$A$-Mod
$KE_1$
$KE_0$
$KE_n$
$\mathscr{H}_T$
$\mathscr{H}_T[-1]$
$\mathscr{H}_T[-n]$
$\mathscr{D}(A)$

Given two $t$-structures $\mathscr{R}$ and $\mathscr{S}$, if

$$\mathscr{R}^{\leq -1} \subseteq \mathscr{S}^{\leq 0} \subseteq \mathscr{R}^{\leq 0} \quad \text{and} \quad \mathscr{R}^{\geq 0} \subseteq \mathscr{S}^{\geq 0} \subseteq \mathscr{R}^{\geq -1}$$

then [Polishchuk, 2007]

$$\mathscr{X} = \mathscr{H}_{\mathscr{R}} \cap \mathscr{H}_{\mathscr{S}} \qquad \mathscr{Y} = \mathscr{H}_{\mathscr{R}} \cap \mathscr{H}_{\mathscr{S}}[-1]$$

form a torsion pair $(\mathscr{X}, \mathscr{Y})$ in $\mathscr{H}_{\mathscr{R}}$.

$$\mathscr{D}^{\leq -n} \subseteq \mathscr{T}^{\leq 0} \subseteq \mathscr{D}^{\leq 0} \quad \text{and} \quad \mathscr{D}^{\geq 0} \subseteq \mathscr{T}^{\geq 0} \subseteq \mathscr{D}^{\geq -n}$$

$$\mathscr{D}_0^{\geq 0} := \mathscr{D}^{\geq 0} = \mathscr{D}^{\geq 0} \cap \mathscr{T}^{\geq 0}$$

$$\mathscr{D}_1^{\geq 0} := \mathscr{D}^{\geq -1} \cap \mathscr{T}^{\geq 0}$$

$$\mathscr{D}_2^{\geq 0} := \mathscr{D}^{\geq -2} \cap \mathscr{T}^{\geq 0}$$

$$\ldots$$

$$\mathscr{D}_n^{\geq 0} := \mathscr{D}^{\geq -n} \cap \mathscr{T}^{\geq 0} = \mathscr{T}^{\geq 0}$$

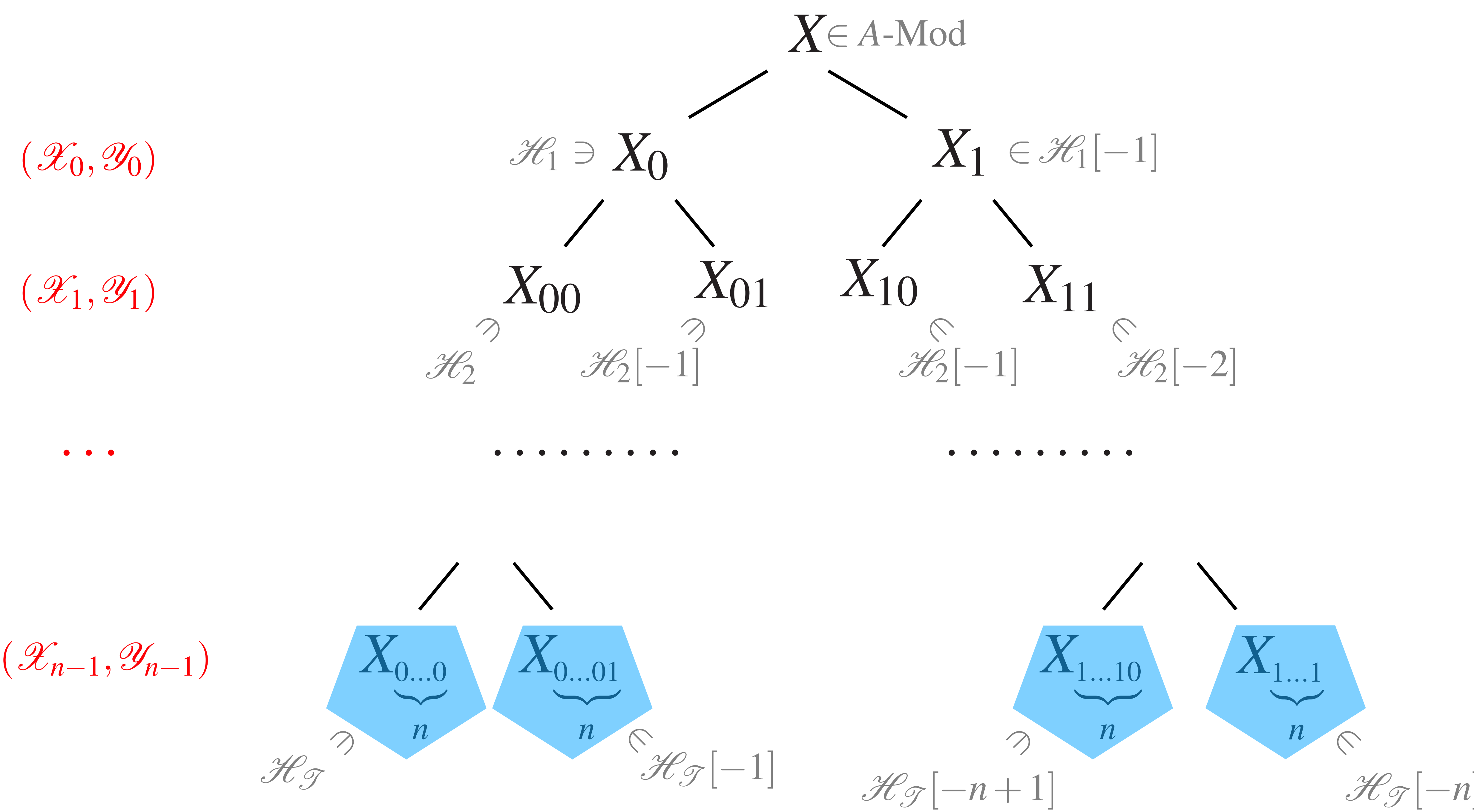
$\mathscr{D}_i^{\geq 0} = \mathscr{D}^{\geq -i} \cap \mathscr{T}^{\geq 0}$ are coaisles for $i = 0, \ldots n$!

Denote by $\mathscr{H}_i$ the heart of the $t$-structure $\mathscr{D}_i = (\mathscr{D}_i^{\leq 0}, \mathscr{D}_i^{\geq 0})$.

$$\left(\mathscr{X}_i := \mathscr{H}_i \cap \mathscr{H}_{i+1},\ \mathscr{Y}_i := \mathscr{H}_i \cap \mathscr{H}_{i+1}[-1]\right)$$

is a torsion pair in $\mathscr{H}_i$.

# The $t$-tree



$X \in A$-Mod
$(\mathscr{X}_0, \mathscr{Y}_0)$
$\mathscr{H}_1 \ni X_0$
$X_1 \in \mathscr{H}_1[-1]$
$(\mathscr{X}_1, \mathscr{Y}_1)$
$X_{00}$
$X_{01}$
$X_{10}$
$X_{11}$
$\mathscr{H}_2$
$\mathscr{H}_2[-1]$
$\mathscr{H}_2[-1]$
$\mathscr{H}_2[-2]$
$(\mathscr{X}_{n-1}, \mathscr{Y}_{n-1})$
$X_{\underbrace{0\ldots0}_{n}}$
$X_{\underbrace{0\ldots01}_{n}}$
$X_{\underbrace{1\ldots10}_{n}}$
$X_{\underbrace{1\ldots1}_{n}}$
$\mathscr{H}_{\mathscr{T}}$
$\mathscr{H}_{\mathscr{T}}[-1]$
$\mathscr{H}_{\mathscr{T}}[-n+1]$
$\mathscr{H}_{\mathscr{T}}[-n]$

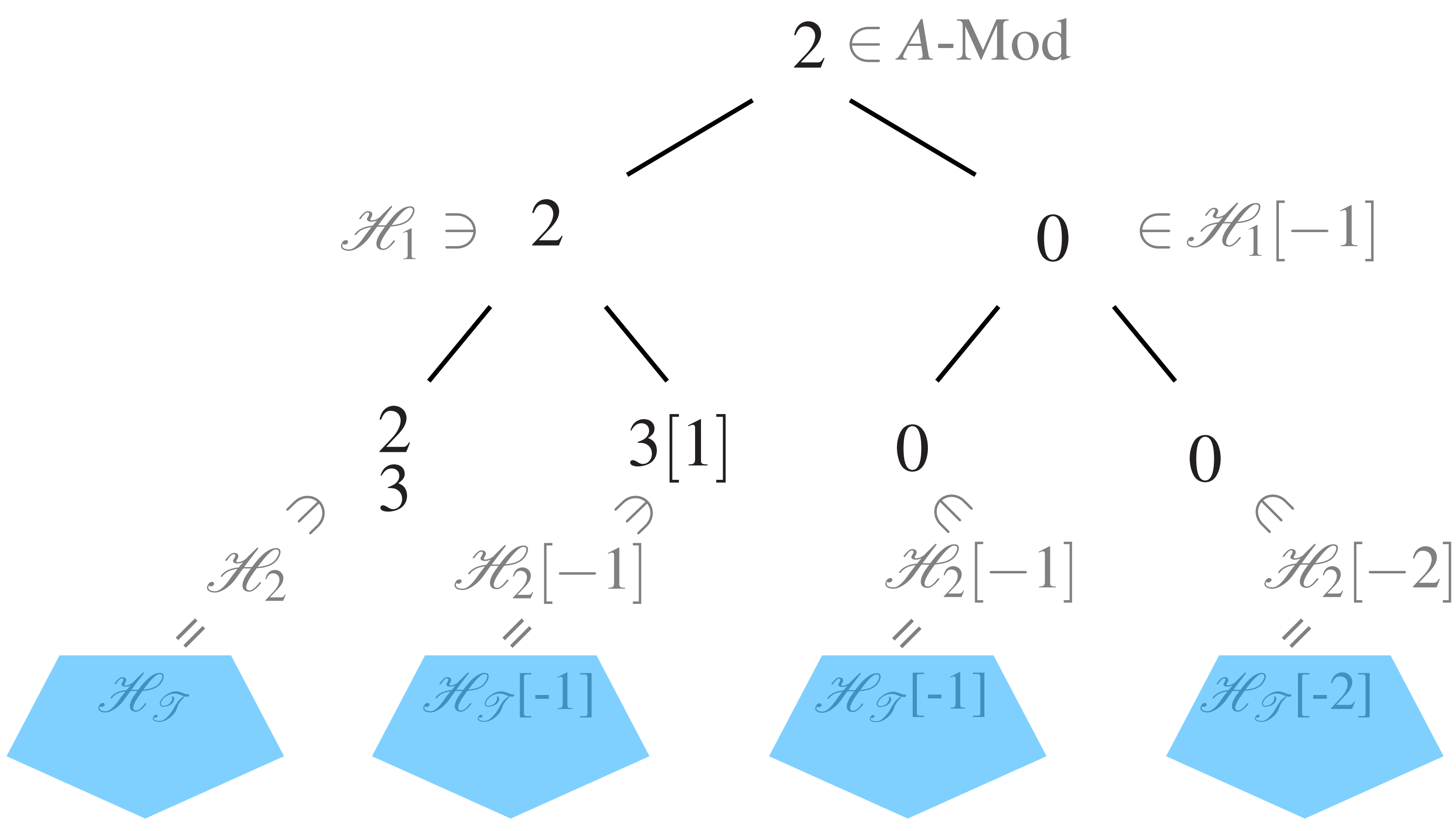
$A$= $k$-algebra associated to $1 \xrightarrow{a} 2 \xrightarrow{b} 3$ with $b \circ a = 0$.

$T = \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1$ is a 2-tilting $A$-module.

$\mathrm{Hom}_A(T, 2) \simeq \mathrm{Ext}^1_A(T, 2) \neq 0 \Longrightarrow 2 \notin \mathrm{KE}_\ell, \ell = 0, 1, 2.$

$(\mathscr{X}_0, \mathscr{Y}_0) = \left(\{1, 2, \begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}2\\3\end{smallmatrix}\}, \{3\}\right)$

$(\mathscr{X}_1, \mathscr{Y}_1) = \left(\{1, \begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}2\\3\end{smallmatrix}, \begin{smallmatrix}2\\3\end{smallmatrix} \to \begin{smallmatrix}\dot{1}\\2\end{smallmatrix}\}, \{3[1]\}\right)$



Grazie per l’attenzione!