Unsettled linear algebra of Fourier transforms of complex-coefficient pseudo-Gaussians.

David Milovich
http://dkmj.org

Welkin Sciences
Colorado Springs, CO

UCCS Algebra Seminar
Mar. 8, 2023

Table of Contents

Linear Channels

My job revolves around statistical models of random linear channels. Linear channels work like this:

- A complex tone $s(t) = e^{2\pi i f t}$ of frequency f is trasmitted.
- At position (x, y) on a receiving antenna aperture, the received signal is

$$r(t) = T(x, y, f, t)e^{2\pi i f t}.$$

- T is called the transfer function.
- More realistically, the transmitted complex signal is a continuous linear combination of tones

$$s(t) = \int_{\mathbb{R}} S(f)e^{2\pi i f t} df$$

and the total received complex signal is

$$r(t) = \iiint_{\mathbb{R}^3} A(x, y)S(f)T(x, y, f, t)e^{2\pi i f t} dx\, dy\, df$$

for some antenna aperture weight function $A(x, y)$.

Random Linear Channels

- The transfer function T is modelled as a complex-valued random process drawn from a known distribution.
- Modulo some strong assumptions, the distribution of T is characterized by an autocovariance function

$$R(\Delta x, \Delta y, \Delta f, \Delta t) = \left\langle \overline{T(x, y, f, t)} T(x + \Delta x, y + \Delta y, f + \Delta f, t + \Delta t) \right\rangle$$

- **Notation:** $\langle \bullet \rangle$ denotes mean value.
- **Notation:** $\overline{\bullet}$ denotes complex conjugation.
- When I can't get away with such strong assumptions, I have to think about higher-order moments like

$$\left\langle \overline{T(x_1, y_1, f_1, t_1,)} T(x_2, y_2, f_2, t_2,) \overline{T(x_3, y_3, f_3, t_3,)} T(x_4, y_4, f_4, t_4) \right\rangle .$$

Gaussian Functions

- Call a function $g\colon \mathbb{R}^n \to \mathbb{R}$ Gaussian if

$$g(\mathbf{x}) = g(\mathbf{0}) \exp\left(-\frac{1}{2}\mathbf{x}^T A\,\mathbf{x}\right)$$

  for some real symmetric positive-definite $n \times n$ matrix A.
- Symmetric means the transpose A^T equals A.
  There is no loss of generality is assuming A is symmetric.
- Positive-definite means $\mathbf{x}^T A\,\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.

Gaussian Channel Models

- Because of the Central Limit Theorem, Gaussian autocovariance functions arise naturally.
- A simple example:

$$
\begin{aligned}
R_1(\Delta x, \Delta y, \ldots) &= \exp\left(-\frac{1}{2}\begin{bmatrix}\Delta x & \Delta y\end{bmatrix}\begin{bmatrix}L_x^{-2} & 0 \\ 0 & L_y^{-2}\end{bmatrix}\begin{bmatrix}\Delta x \\ \Delta y\end{bmatrix} - \cdots\right) \\
&= \exp\left(-\frac{\Delta x^2}{2L_x^2} - \frac{\Delta y^2}{2L_y^2} - \cdots\right)
\end{aligned}
$$

  The parameters L_x and L_y are <u>decorrelation lengths</u>.
- R_1 is (part of) a decent model for the autocovariance of the warping of the wavefront of a microwave transmission that just passed through the ionosphere, which has (practically) random stripes of higher and lower indices of refraction.

Pseudo-Gaussian Functions

- Call a function $g\colon \mathbb{R}^n \to \mathbb{C}$ pseudo-Gaussian if

$$g(\mathbf{x}) = g(\mathbf{0}) \exp\left(-\frac{1}{2}\mathbf{x}^T A\,\mathbf{x}\right)$$

  for some complex symmetric $n \times n$ matrix A with positive-definite real part $\Re A$.

- A pseudo-Gaussian is a Gaussian multiplied by a complex oscillating function.

$$|g(\mathbf{x})| = |g(\mathbf{0})| \exp\left(-\frac{1}{2}\mathbf{x}^T (\Re A)\,\mathbf{x}\right)$$

$$\arg(g(\mathbf{x})) = \arg(g(\mathbf{0})) - \frac{1}{2}\arg(\mathbf{x}^T (\Im A)\,\mathbf{x})$$

- Why can't we assume A is Hermitian ($A^T = \overline{A}$)?
  A with a non-real diagonal is an important case.

Pseudo-Gaussian Channel Models

- After that microwave passes through the ionosphere, diffraction occurs: the warps in the wavefront interfere with each other as they propagate through space or through the lower atmosphere.
- To model this diffraction, autocovariance function R is evolved from an initial value $R(z = 0, \Delta x, \ldots) = R_1(\Delta x, \ldots)$ according to a differential equation like this:

$$0 = \left(\frac{\partial}{\partial z} + \frac{b}{2i} \Delta f \frac{\partial^2}{\partial (\Delta x)^2} + \cdots \right) R(z, \ldots).$$

  b is some positive (real) constant.
- The final value $R(z = L, \ldots) = R_2(\ldots)$ is a pseudo-Gaussian with respect to $(\Delta x, \Delta y, \Delta t)$:

$$R_2(\Delta x, \ldots) = \frac{L_x}{\sqrt{L_x^2 + ibL\Delta f}} \exp\left(-\frac{\Delta x^2}{2(L_x^2 + ibL\Delta f)} - \cdots \right)$$

Fourier Transforms

- The formula for R_2 was found using Fourier transforms.
- A function $s\colon \mathbb{R}^n \to \mathbb{C}$ is Schwartz if all its partial derivatives exist everywhere and decay superpolynomially:

$$\lim_{|\mathbf{x}|\to\infty} |\mathbf{x}|^M \frac{\partial^N s(\mathbf{x})}{\partial x_{k_1} \cdots \partial x_{k_N}} = 0$$

  for all M, N, and $k_1, \ldots, k_N$.
- Pseudo-Gaussians are Schwartz.
- Define the Fourier transform $\mathcal{F}s\colon \mathbb{R}^n \to \mathbb{C}$ of s by

$$\mathcal{F}s(\mathbf{y}) = \int_{\mathbb{R}^n} \exp(-i\mathbf{y}^T\mathbf{x}) s(\mathbf{x}) d\mathbf{x}.$$

- Lemma. The Fourier transform of a Schwartz function is another Schwartz function.

Pseudo-Gaussians Transformed

- Main Lemma. If a complex symmetrix matrix A has positive-definite real part, then all eigenvalues of A have positive real parts. In particular, A is invertible.
- Theorem. The Fourier transform of pseudo-Gaussian g is another pseudo-Gaussian.
- Proof.
  1. $g(\mathbf{x}) = g(\mathbf{0}) \exp(-\mathbf{x}^T A \mathbf{x}/2)$.
  2. $\nabla g(\mathbf{x}) = -g(\mathbf{x}) A \mathbf{x}$.
  3. $i\mathcal{F}g(\mathbf{y})\mathbf{y} = -iA\nabla\mathcal{F}g(\mathbf{y})$.
  4. $-\mathcal{F}g(\mathbf{y})A^{-1}\mathbf{y} = \nabla\mathcal{F}g(\mathbf{y})$.
  5. $\mathcal{F}g(\mathbf{y}) = \mathcal{F}g(\mathbf{0}) \exp(-\mathbf{y}^T A^{-1} \mathbf{y}/2)$.
  6. Because $\mathcal{F}g$ is Schwartz, $\Re(A^{-1})$ must be positive-definite.
- Corollary. If $A^T = A$ and $\Re A$ is positive-definite, then $\Re(A^{-1})$ is also positive-definite.
- But what is $\mathcal{F}g(\mathbf{0})$?

$$\mathcal{F}g(\mathbf{0}) = g(\mathbf{0}) \int_{\mathbb{R}^n} \exp(-\mathbf{x}^T A \mathbf{x}/2) d\mathbf{x} = \ ?$$

Pseudo-Gaussian Integrals Motivated

Again, given $A^T = A$ and $\Re A$ is positive-definite,

$$\int_{\mathbb{R}^n} \exp(-\mathbf{x}^T A \mathbf{x}/2) d\mathbf{x} = ?$$

- If A is diagonal, the solution is well-known.
- If A is block diagonal with $2x2$ blocks, the solution is still not too hard.
- So far, these cases have sufficed for my channel model work.
- But mathematicians love to generalize!
- Also, future work involving higher-order moments may require the general case.

Table of Contents

Proof of the Main Lemma

1. Assume $A^T = A$ and $\mathbf{x}^T(\Re A)\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$.
2. Assume $\mathbf{v} \in \mathbb{C}^n$, $A\mathbf{v} = \lambda\mathbf{v}$, and $\mathbf{v}^T\overline{\mathbf{v}} = 1$.
3. Use a clever formula for $\Re\lambda$:

$$
\begin{aligned}
0 &< (\Re\mathbf{v})^T(\Re A)(\Re\mathbf{v}) + (\Im\mathbf{v})^T(\Re A)(\Im\mathbf{v}) \\
&= \frac{1}{8}\sum_{+,-}(\mathbf{v} \pm \overline{\mathbf{v}})(A + \overline{A})(\overline{\mathbf{v}} \pm \mathbf{v}) \\
&= \frac{1}{4}\sum_{+,-}\left(\lambda + \overline{\lambda} \pm \Re(\lambda\mathbf{v}^T\mathbf{v}) \pm \Re(\mathbf{v}^T\overline{A}\mathbf{v})\right) \\
&= \Re\lambda
\end{aligned}
$$

Table of Contents

The Real Case

- Theorem. If A is an $n \times n$ real symmetric positive-definite matrix, then

$$\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\mathbf{x}^T A\,\mathbf{x}\right) d\mathbf{x} = \frac{(2\pi)^{n/2}}{\sqrt{\det(A)}} > 0.$$

- Proof.
  1. By the spectral theorem, $A = QDQ^T$ where Q is orthogonal and D is positive diagonal.
  2. Rotating and rescaling via $x = QD^{-1/2}y$, we have

$$\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\mathbf{x}^T A\,\mathbf{x}\right) d\mathbf{x} = \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\mathbf{y}^T\mathbf{y}\right) \frac{d\mathbf{y}}{\sqrt{\det(D)}} = \frac{(2\pi)^{n/2}}{\sqrt{\det(A)}}.$$

The One-Dimensional Complex Case

- Theorem. If $\Re a > 0$, then

$$\int_{\mathbb{R}} \exp\left(-\frac{1}{2}ax^2\right) dx = \frac{\sqrt{2\pi}}{\sqrt{a}}$$

where $\Re\sqrt{a} > 0$.

- Proof.
  1. Let $a = re^{i\phi}$ with $|\phi| < \pi/2$.
  2. The integral $\exp(-rz^2/2)dz$ along the circular arc $z = Re^{it}$ from $t = 0$ to $t = \phi/2$ goes to zero in the limit $R \to \infty$.
  3. Therefore, the Cauchy integral theorem lets us substitute $x = y/e^{i\phi/2}$ and then rotate the domain of integration back to the real line.

$$\int_{\mathbb{R}} \exp\left(-\frac{1}{2}ax^2\right) dx = \int_{\mathbb{R}} \exp\left(-\frac{1}{2}ry^2\right) \frac{dy}{e^{i\phi/2}} = \frac{\sqrt{2\pi}}{\sqrt{r}e^{i\phi/2}}.$$

Comments on Extending the Proofs

- In the real case, the rotation $\mathbb{R}^n \to \mathbb{R}^n$ via the orthogonal matrix Q preserved the integral.
- In the 1D complex case, the rotation $\mathbb{R} \to e^{i\phi/2}\mathbb{R} \subset \mathbb{C}$ preserved the integral only because $|\phi| < \pi/2$.
- Naively using $\phi + 2\pi$ in place of ϕ erroneously multiplies the integral by a factor of -1.
- For complex symmetric A, the closest thing to the spectral theorem is the Takagi factorization $A = UDU^T$ with U unitary ($\overline{U}U^T = I$) and D real nonnegative diagonal.
- But an arbitrary unitary rotation $\mathbb{R}^n \to U\mathbb{R}^n \subset \mathbb{C}^n$ will not preserve the integral.
- Main Question. Can the Takagi factorization be improved under the additional assumption that $\Re A$ is positive definite?

Complex Diagonal Examples

- Notation. If A is an $n \times n$ complex symmetric matrix and $\Re A$ is positive-definite, then define $c(A)$ by

$$\frac{(2\pi)^{n/2}}{c(A)} = \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\mathbf{x}^T A\,\mathbf{x}\right) d\mathbf{x}$$

- Theorem. If $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and $\Re\lambda_k > 0$ for all k, then $c(D) = \prod_k \sqrt{\lambda_k}$ where $\Re\sqrt{\lambda_k} > 0$.
- $c(D)$ is a square root of $\det(D)$ that depends on D, not just on $\det(D)$:
    1. $\det(\mathrm{diag}(e^{2\pi i/5}, e^{2\pi i/5}, e^{2\pi i/5})) = e^{6\pi i/5}$.
    2. $c(\mathrm{diag}(e^{2\pi i/5}, e^{2\pi i/5}, e^{2\pi i/5})) = e^{3\pi i/5}$.
    3. $\det(\mathrm{diag}(e^{-2\pi i/5}, e^{-2\pi i/5}, 1)) = e^{-4\pi i/5} = e^{6\pi i/5}$.
    4. $c(\mathrm{diag}(e^{-2\pi i/5}, e^{-2\pi i/5}, 1)) = e^{-2\pi i/5} = -e^{3\pi i/5}$.

Table of Contents

Main Theorem

Assume A is an $n \times n$ complex symmetric matrix and $\Re A$ is positive-definite.

- Notation.

$$\frac{(2\pi)^{n/2}}{c(A)} = \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\mathbf{x}^T A\,\mathbf{x}\right) d\mathbf{x}$$

- Theorem. $c(A) = \prod_k \sqrt{\lambda_k}$ where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A and $\Re\sqrt{\lambda_k} > 0$.
- Corollary. $c(A)^2 = \det(A)$.
- I've numerically checked that the theorem holds for 10^4 random choices for A of size $n \times n$, for each $n \leq 10$.

Proof of Main Theorem

1. The theorem holds for real symmetric matrices.
2. We can continuously deform $\Re A$ to A by the straight-line homotopy $A(t) = \Re A + t\Im A$, which preserves our hypotheses $A^T = A$ and $\Re A$ positive-definite.
3. By the main lemma, $\det(A) \neq 0$.
4. Because $c(A)^2 = \left(\prod_k \sqrt{\lambda_k}\right)^2 = \det(A) \neq 0$, if both $c(A)$ and $\prod_k \sqrt{\lambda_k}$ are continuous functions of A, then the homotopy will preserve $c(A) = \prod_k \sqrt{\lambda_k}$.
5. $A \mapsto c(A)$ is a continuous because of its integral formula.
6. $\prod_k \sqrt{\lambda_k}$ is a continuous function of the unordered n-tuple $\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}$.
7. $z \to \sqrt{z}$ with $\Re\sqrt{z} > 0$ is continuous on $\{z \in \mathbb{C} \mid \Re z > 0\}$.
8. By the main lemma, $\Re\lambda_k > 0$.
9. The unordered n-tuple $\lambda_1, \ldots, \lambda_n$ is a continuous function of of the coefficients of $\det(\lambda I - A)$, which in turn are continuous functions of A.

Comments on the Proof

- The proof is a continuity argument.
- A more algebraic proof would yield more algebraic insight.
- A more algebraic proof might lead to an improved Takagi factorization.

Table of Contents

Motivation

- We know that $c(A) = \prod_k \sqrt{\lambda_k}$.
- But that is not the only way to compute $c(A)$.
- Other algorithms might turn out to be helpful for finding a more algebraic proof of the main theorem.
- At least one other algorithm is more efficient.

A Direct Computation of $c(A)$

1. Let $A = P + iS$ where P and S are real symmetric $n \times n$ matrices and P is positive definite.
2. By the spectral theorem, $P = QDQ^T$ where D is positive diagonal and Q is orthogonal.
3. Rotating and rescaling via $x = QD^{-1/2}y$, we have

$$\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\mathbf{x}^T A\,\mathbf{x}\right) d\mathbf{x} = \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\mathbf{y}^T(I + iB)\mathbf{y}\right) \frac{d\mathbf{y}}{\sqrt{\det(D)}}$$

   where $B = D^{-1/2}Q^T SQD^{-1/2}$.
4. By the spectral theorem again, $B = VEV^T$ where V is orthogonal and E is real diagonal.

A Direct Computation of $c(A)$ (continued)

5. Rotating via $y = Vz$, we have

$$\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\mathbf{y}^T(I + iB)\mathbf{y}\right)\frac{d\mathbf{y}}{\sqrt{\det(D)}}$$
$$= \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\mathbf{z}^T(I + iE)\mathbf{z}\right)\frac{d\mathbf{z}}{\sqrt{\det(D)}}$$
$$= \frac{1}{\sqrt{\det(D)}}\prod_k \frac{\sqrt{2\pi}}{\sqrt{1 + i\mu_k}}$$

where $E = \mathrm{diag}(\mu_1, \ldots, \mu_n)$ and $\Re\sqrt{1 + i\mu_k} > 0$.

6. $c(A) = \sqrt{\det(D)}\prod_k \sqrt{1 + i\mu_k}$.

Thus, $c(A)^2 = \det(D)\det(I + iE) = \det(P)\det(I + iB) = \det(A)$.

A Cholesky-Type Block Factorization

Assume A is a complex symmetric matrix, that $\Re A$ is positive-definite, and that A divides into blocks as follows.

$$A = \begin{bmatrix} E & F \\ F^T & G \end{bmatrix}$$

- If E is invertible, then

$$A = \begin{bmatrix} I & 0 \\ F^T E^{-1} & I \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & G - F^T E^{-1} F \end{bmatrix} \begin{bmatrix} I & E^{-1}F \\ 0 & I \end{bmatrix}$$

  and, hence, $\det(A) = \det(E)\det(G - F^T E^{-1} F)$.

- Lemma. E is invertible. Moreover, $\Re E$ and $\Re(G - F^T E^{-1} F)$ are positive-definite.

Divide and Conquer

Assume $A = \begin{bmatrix} E & F \\ F^T & G \end{bmatrix}$ is an $n \times n$ complex symmetric matrix and $\Re A$ is positive-definite.

- Notation. Given $\mathbf{b} \in \mathbb{C}^n$, define

$$\frac{(2\pi)^{n/2}}{c(A, \mathbf{b})} = \int_{\mathbb{R}^n + \mathbf{b}} \exp\left(-\frac{1}{2}\mathbf{x}^T A\,\mathbf{x}\right) d\mathbf{x}.$$

- Lemma. $c(A, \mathbf{b}) = c(E, \mathbf{0})c(G - F^T E^{-1} F, \mathbf{0})$.
- Proof. Use induction on n and the change of variables

$$\begin{bmatrix} \mathbf{x}' \\ \mathbf{y}' \end{bmatrix} = \begin{bmatrix} \mathbf{x} + E^{-1}F\mathbf{y} \\ \mathbf{y} \end{bmatrix}$$

corresponding to the factorization

$$A = \begin{bmatrix} I & 0 \\ F^T E^{-1} & I \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & G - F^T E^{-1} F \end{bmatrix} \begin{bmatrix} I & E^{-1}F \\ 0 & I \end{bmatrix}.$$

Again a Square Root of the Determinant

Assume $A = \begin{bmatrix} E & F \\ F^T & G \end{bmatrix}$ is an $n \times n$ complex symmetric matrix and $\Re A$ is positive-definite.

- Notation.

$$\frac{(2\pi)^{n/2}}{c(A)} = \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\mathbf{x}^T A\,\mathbf{x}\right) d\mathbf{x}$$

- Alternative proof of $c(A)^2 = \det(A)$:
  1. Proceed by induction on n.
  2. We already proved case $n = 1$.
  3. For $n > 1$, use $c(A) = c(E)c(G - F^T E^{-1} F)$ and $\det(A) = \det(E)\det(G - F^T E^{-1} F)$.

Table of Contents

Open Problems

1. Find a more algebraic proof that $c(A) = \prod_k \sqrt{\lambda_k}$ with $\Re\sqrt{\lambda_k} > 0$.
2. Find an improvement of the Takagi factorization $A = UDU^T$ of complex symmetric matrices under the assumption that $\Re A$ is positive-definite.