On Naimark's Problem for Graph Algebras

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Rings and Wings Seminar

Dedicated to Professor Laszlo Fuchs on his 99th Birthday

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- We say two representations $\lambda : A \to B(\mathcal{H})$ and $\mu : A \to B(\mathcal{H})$ are **unitary equivalent** if there is a unitary operator $u : \mathcal{H} \to \mathcal{H}$ such that $u^*\lambda(a)u = \mu(a)$ for all $a \in A$. Here, u is **unitary** means that $u^*u = uu^* = 1$. Unitary equivalence is an equivalence relation.

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- If two representations $\lambda : A \to B(\mathcal{H})$ and $\mu : A \to B(\mathcal{H})$ are unitary equivalent, then $\ker(\lambda) = \ker(\mu)$.

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- In 1948, **Naimark** proved that the C*-algebra $\mathcal{K}(\mathcal{H})$ of compact operators on a Hilbert space $\mathcal{H}$ possesses only one irreducible representation up to unitary equivalence.
- **Naimark's Question**: Should any C*-algebra A with the above property be isomorphic to $\mathcal{K}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$?

History

- (1951) (**Kaplansky**): The answer to Naimark's question is YES, if A is a Type I C*-algebra (If $\lambda : A \to B(\mathcal{H})$ is any irreducible representation, then $\lambda(A) \supseteq \mathcal{K}(\mathcal{H})$).

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- Partial solutions by several researchers such as **Diximier** (1960), **Fell** (1961), **Glimm** (1961). (separable + unique irred.repsn = Type I).
- For quite sometime, it was not clear whether Naimark's problem for uncountable C*-algebras has a solution or not.

- After more than 40 years, in 2004 Akemann and Weaver used Jensen's **diamond axiom** (a combinatorial principle independent of ZFC) to provide a negative solution to Naimark's problem by constructing an $\aleph_1$-generated C*-algebra with Naimark's property, but not isomorphic to $\mathcal{K}(\mathcal{H})$. They also showed that it is undecidable in ZFC whether there exists an $\aleph_1$-generated C*-algebra with Naimark's property, but not isomorphic to $\mathcal{K}(\mathcal{H})$.

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- Not much progress was made until 2017, when **N. Suri and M. Tomforde** considered the Naimark's problem in the context of graph C*-algebras and showed that Naimark's problem has a positive solution for special type of graph C*-algebras $C^*(E)$ called AF algebras and also when E is countable.

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- In this talk, instead of considering special cases, we will use graphical techniques to directly solve Naimark's problem for arbitrary graph C*-algebras.

Graph Preliminaries

- A (directed) graph $E = (E^0, E^1, r, s)$ consists of two sets E^0 and E^1 together with maps $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*.

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- A vertex v is called a **sink** if it emits no edges and a vertex v is called an **infinite emitter** if it emits infinitely many edges. A **regular vertex** is a vertex which emits a non-empty finite set of edges. A vertex which is an infinite emitter or a sink is called a **singular vertex**.

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- A **path** is either a vertex or a finite sequence of edges $\mu = e_1 e_2 \cdots e_n$ with $n \geq 1$, where $r(e_i) = s(e_{i+1})$ for all $i = 1, \cdots, n-1$. The set of all vertices on the path μ is denoted by μ^0.

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- A path $\mu = e_1 \ldots e_n$ in E is **closed** if $r(e_n) = s(e_1)$, in which case μ is said to be *based at the vertex* $s(e_1)$. The closed path μ is called a **cycle** if it does not pass through any of its vertices twice, that is, if $s(e_i) \neq s(e_j)$ for every $i \neq j$.

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- A subset H of E^0 is called **hereditary** if, whenever $v \in H$ and $w \in E^0$ satisfy $v \geq w$, then $w \in H$. A hereditary set H is **saturated** if, for any regular vertex v, $r(s^{-1}(v)) \subseteq H$ implies $v \in H$

- Let $E = (E^0, E^1, r, s)$ be an arbitrary graph. The **graph C*-algebra** $C^*(E)$ is the universal C*-algebra generated by mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ with mutually orthogonal ranges called Cuntz-Krieger family, satisfying

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- (2) $s_e s_e^* \leq p_{s(e)}$ for all $e \in E^1$
- (3) $p_v = \sum_{e \in s^{-1}(v)} s_e s_e^*$, for any regular vertex v.

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- **Known**: Every proper closed ideal (including $\{0\}$) of a C*-algebra is the intersection of primitive ideals.
- **Proposition 1**: *If a C*-algebra A has exactly one equivalence class of irreducible representations, then A must be a simple algebra*.
- **Proof**: Since there is only one irreducible representation up to equivalence and since equivalent representations have the same kernel, A will then have only one primitive ideal P. But any proper closed ideal I of A is the intersection of primitive ideals of A and so $I = P$. In particular, the zero ideal $\{0\} = P = I$. Thus A is simple.

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- **Theorem 4**: (A.L.T. Paterson-W. Szymanski) *$C^*(E)$ is simple if and only if the graph E is downward directed, contains no proper non-empty hereditary saturated subset of vertices, and $u \geq w$ for every vertex u and every singular vertex w.*

- In this talk, I first give characterizing conditions on the graph E under which $C^*(E) \cong \mathcal{K}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. The graphical approach enables us to obtain a streamlined proof that Naimark's question has a positive answer for arbitrary graph C*-algebras $C^*(E)$.

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- Also, we characterize $C^*(E)$ having finite or countable infinite number of irreducible representations (up to unitary equivalence).
- If time permits, we will state and prove the algebraic version of Naimark's problem for Leavitt path algebras.
- A perhaps interesting conclusion is: Given a graph E, Naimark's problem has a positive solution for $C^*(E) \Leftrightarrow$ Naimark's problem (algebraic version) has a positive solution for $L_K(E)$.

- Two infinite paths $p = e_1 e_2 \cdots e_n \cdots$ and $q = f_1 f_2 \cdots f_n \cdots$ in a graph E are said to be **shift-tail equivalent** or **tail-equivalent**, if there exist positive integers m, n such that $e_{m+i} = f_{n+i}$ for all $i \geq 0$.

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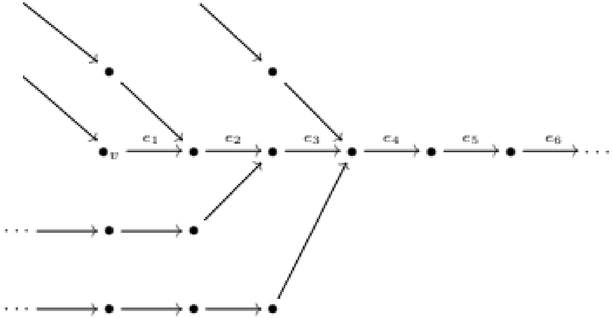
- 

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- As we shall see, tail equivalent classes $[p]$ of infinite paths p give rise to irreducible representations of $C^*(E)$ similar to the way irreducible representations are constructed for Leavitt path algebras.

Constructing irreducible representations

Theorem 5:(T.M. Carlsen and A. Sims - 2019) *Let E be an arbitrary graph. For each tail-equivalent class $[p]$ of infinite paths in E and each $z \in \mathbb{T}$* (the unit circle in the complex plane), *there is an irreducible representation $\pi_{p,z} : C^*(E) \longrightarrow B(\ell^2([p]))$ such that, for all $q \in [p], v \in E^0$ and $e \in E^1$,*
$\pi_{p,z}(p_v)(q) = q$ or 0 according as $v = s(q)$ or not ;
$\pi_{p,z}(s_e)(q) = zeq$ or 0 according as $r(e) = s(q)$ or not.
If $z, t \in \mathbb{T}$ with $z \neq t$ and p, q are infinite paths with $[p] \neq [q]$, then $\pi_{p,z}$ and $\pi_{q,t}$ are not unitary equivalent.
"Distinct equivalence classes of infinite paths in E yield distinct inequivalent irreducible representations for $C^*(E)$"

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- We say a vertex v has *bifurcation* or v is a **bifurcation vertex** if v emits two or more edges.
- A vertex v is called a **line point** if $T(v)$ contains no bifurcating vertices and no cycles. Thus $T(v)$ becomes, when we add all the edges between any two vertices in $T(v)$, a straight line path like $\cdot_{v=v_1} \to \cdot_{v_2} \to \cdot_{v_3} \cdot \to \cdots$ which becomes a finite path if $T(v)$ contains a sink. In particular, a sink itself is a line point.

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- If $\{v_i : i \in I\}$ is an orthonormal basis of $\mathcal{H}$, then $\mathcal{K}(\mathcal{H})$ is generated by a set of **matrix units** R_{ij}, $(i, j \in I)$, that is, R_{ij} satisfy for all $i, j, k, l \in I$, $R_{ij}R_{kl} = R_{il}$ or 0 according as $j = k$ or not and, further, $R_{ij}^* = R_{ji}$. Here, R_{ij} is a rank-1 operator on $\mathcal{H}$ given by $R_{ij}(h) =< h, v_j > v_i$ for all $i, j \in I$ and $h \in \mathcal{H}$.

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- **Theorem 6**: (Raeburn, Corollary A-9 and Remark A-10 [5]) If a C*-algebra $A \neq 0$ is generated by a set of non-zero matrix units, then $A \cong \mathcal{K}(\mathcal{H})$ for a suitable $\mathcal{H}$.

- **Theorem 7:** *Let E be an arbitrary graph. Then a graph C*-algebra $A = C^*(E)$ is isomorphic to $\mathcal{K}(\mathcal{H})$ if and only if E contains no cycles, no infinite emitters, the vertex set E^0 is downward directed and is the hereditary saturated closure of a line point v.*

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- Case 2: v is a line point, not a sink, with $T(v) =$ $\cdot_{v=v_1} \longrightarrow \cdot_{v_2} \longrightarrow \cdot_{v_3} \longrightarrow \cdot_{v_4} \longrightarrow \cdots$. For each $i, j \in \mathbb{N}$, let p_{ij} be the unique path connecting v_i to v_j. Define $x_{ij} = p_{ij}$ or p_{ij}^* according as $i \leq j$ or $i \geq j$. Then $C^*(E)$ is generated by the set of matrix units $\{s_{\alpha_i} s_{x_{ij}} s_{\beta_j} : \alpha_i, \beta_j \text{ paths in } E, r(\alpha_i) = v_i, r(\beta_j) = v_j, i, j \in \mathbb{N}$ and α_i, β_j contain no other vertices $v_k \in T(v)\}$.

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- In both cases, by Raeburn's Theorem (Theorem 6), $C^*(E) \cong \mathcal{K}(\mathcal{H})$ for a suitable Hilbert space $\mathcal{H}$.

- **Remark**: From the proof of Theorem 7, it is clear that there are only two types of graphs E for which $C^*(E) \cong \mathcal{K}(\mathcal{H})$.

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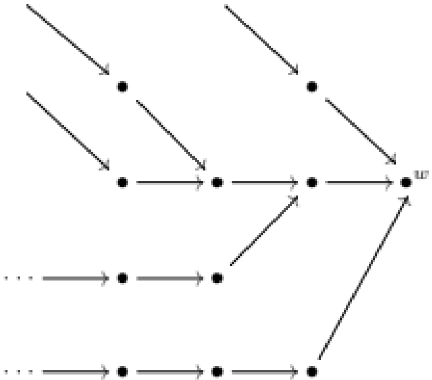
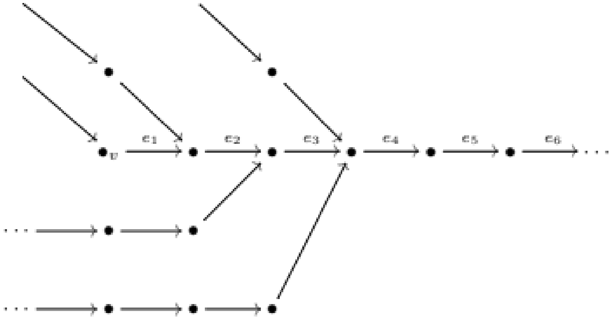
- E is a row-finite acyclic graph which is
- **Either** (i) a comet with a sink as its head (Example 1);
- **Or** (ii) is just a tail-equivalent class $[p]$ of an infinite path p with $s(p)$ a line point. (Example 2)

Example 1: "comet like" graph



Example 2:



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- Using Carlsen-Sims theorem (Theorem 5), we then get the following:
- **Corollary 9**: *If a graph E satisfies one of the conditions in the above theorem, then $C^*(E)$ possesses uncountably many irreducible representations no two of which is unitary equivalent.*

Theorem

Let E be an arbitrary graph. TFAE for $A = C^*(E)$:
(a) A has, up to unitary equivalence, exactly one irreducible representation;
(b) E is a row-finite acyclic graph such that the vertex set E^0 is downward directed and is the hereditary saturated closure of a single line point v.
(c) $A \cong K(\mathcal{H})$, the C-algebra of compact operators on a suitable Hilbert space $\mathcal{H}$.*

Corollary

If a graph C-algebra A has, up to unitary equivalence, exactly one irreducible representation, then A must be an AF-algebra.*

Proof.

Enough to show (a) $=>$ (b). Assume (a). By Proposition 1, $A = C^*(E)$ is simple and by Theorem 5, (Pattterson-Szimanski), the graph E is then downward directed, satisfies Condition (K), contains no proper non-empty hereditary saturated subset of vertices, and $u \geq w$ for every vertex u and every singular vertex w. If there is a cycle c based at a vertex v, by Condition (K), v is the base of another cycle $d \neq c$ and Corollary 9 will imply that A has uncountably many inequivalent irreducible representations, a contradiction. So E is acyclic. We claim that E contains no infinite emitters. Indeed, if w is an infinite emitter and e is one of the edges emitted by w with $r(e) = u$, then as noted above, $u \geq w$, so there is a path p from u to w. Then ep gives rise to a cycle, a contradiction. Further, Theorem 8 implies that E contains line points. Hence Condition (b) holds. □

C*(E) with countably many irreducible representations

Theorem

Let κ be a finite or countably infinite ordinal. Then $C^*(E)$ has at most κ many unitary equivalence classes of irreducible representations if and only if $C^*(E)$ is an AF-algebra which is the union of a smooth well-ordered ascending chain of guage-invariant ideals indexed by ordinals $\alpha < \kappa$

$$\{0\} = I_0 \subsetneq \cdots \subsetneq I_\alpha \subsetneq I_{\alpha+1} \subsetneq \cdots \qquad (\alpha < \kappa)$$

such that, for each $\alpha < \kappa$, $I_{\alpha+1}/I_\alpha$ is simple and $I_{\alpha+1}/I_\alpha \cong \mathcal{K}(\mathcal{H}_{\alpha+1})$, the algebra of compact operators on a Hilbert space $\mathcal{H}_{\alpha+1}$.

Example 3: Let E be the "Infinite Clock"

$$(\infty) \quad \begin{array}{ccccc} & & \cdot v_1 & & \cdot v_2 \\ & \nwarrow & \uparrow & \nearrow & \\ & & \cdot_v & \longrightarrow & \cdot v_3 \\ & & \downarrow & \searrow & \\ & & \cdot & & \cdot \end{array} .$$

Then $C^*(E)$ has exactly two non-equivalent irreducible representations. If I is the gauge invariant ideal generated by $H = \{v_1, v_2, v_3, ...\}$, then $I \cong \mathcal{K}(\mathcal{H})$ for a suitable Hilbert space $\mathcal{H}$ and $C^*(E)/I \cong \mathbb{C}$.

Example 4: The "Pyramid" graphs enable us to construct for each cardinal κ, finite or infinite, a graph C*-algebra $C^*(E)$ possessing exactly κ many non-equivalent irreducible representations.

Theorem: Let E be an arbitrary directed graph and K be a field. Then the following are equivalent for the Leavitt path algebra $L := L_K(E)$:
(a) Any two simple left/right L-modules are unitary equivalent;
(b) Any two simple left/right L-modules are isomorphic;
(c) E is a row-finite acyclic graph such that the vertex set E^0 is downward directed and is the hereditary saturated closure of a single line point;
(d) $L \cong M_\Lambda(K)$ for some non-empty index set Λ;
(e) L is isomorphic to the algebra of all finite rank linear transformations on a vector space over the field K.

Corollary: Given a graph E, Naimark's problem has a positive solution in $C^*(E) \iff$ Naimark's problem (algebraic version) has a positive solution in $L_K(E)$.

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